Expected Time of Interactive Markov Chains

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Let G be the set of goal states. The expected time to reach G will be calculated as follows

$$\begin{aligned} expTime(s) &= 0, \quad if\ s \in G, \\ expTime(s) &= \frac{1}{E(s)} + \sum_{s' \in S} P(s, s') \cdot expTime(s'), \quad if\ s \notin G, \end{aligned} \tag{1}$$

where $E(s) = \sum_{s' \in S} R(s, s')$ is the exit rate of state s, $P(s, s') = \frac{R(s,s')}{E(s)}$ the probability to move from s to s', and $expTime(s')$ the expected time of state s'. The first part of equation 1 describes the time, that is spent in state s. The second part, the summation, sums up the expected time of all successor states with respect to the probability to move from s to a direct successor s'.

We define the set of predecessors of G

$$Pre(G) = \{s \in S | \exists s' \in G . s \rightarrow^{*} s'\},$$

where $\rightarrow^{*}$ denotes the reflective and transitive closure of interactive and Markovian transition relations. If $s \in Pre(G)$, there exists a way via interactive or Markovian transitions from s to a goal state, otherwise the expected time will be ∞. Further, for all states $s \in G$, the expected time will be 0.

Computing the maximum expected time: In case of an IMC we have to distinguish between Markovian and interactive states. Therefore we use the same idea as in [1]. We compute the expected time with an value iteration algorithm. In each step $i = 0, 1, \ldots, k$ of the iteration, we use two vectors v_i and u_i, where v_i is the expected time vector obtained from u_{i-1} by one step in the classical value iteration algorithm and u_i is obtained by computing the backwards closure along interactive transitions w.r.t. v_{i-1}. Further, we stop the iteration after k steps if we reach a fixpoint such that $u_{i-1} = u_i$ for all $i > k$. We will compute over the set $S' = Pre(G)$. For all states $s \notin S'$ the expected time will be infinity. Further, let $IS' = IS \cap S'$. The first step of the algorithm is to compute v_i such that

$$\forall i \in \{0, \ldots, k\}. v_i(s) = \begin{cases} \frac{1}{E(s)} + \sum_{s' \in S'} P(s, s') \cdot u_{i-1}(s') & \text{if } s \in S' \setminus IS' \setminus G \\ 0 & \text{if } s \in G \\ u_{i-1}(s) & \text{if } s \in IS' \setminus G \end{cases}$$

We use the *reward* vector in the calculation, to exclude deadlock states in the computation. The second step is to compute u_i such that

$$\forall i \in \{0, \dots, k\}.u_i(s) = \begin{cases} max\{v_i(s')|\exists s' \in S'.s \rightsquigarrow_i^* s'\} & \text{if } s \in S' \setminus G \\ 0 & \text{if } s \in G \end{cases}$$

The set $\{s' \in S'|s \rightsquigarrow_i^* s'\}$, where $\rightsquigarrow_i^*$ denotes the reflective and transitive closure of interactive transition relations, can be precomputed by a transitive closure algorithm (which has cubic time complexity).

If we reach a fixpoint after k value iteration steps, $expTime(s)$ from equation (1) is obtained in $u_k(s)$.
In case of the minimum expected time, the calculation will be analogously to the maximum expected time, with the restrictions from the minimum step-bounded reachability from [1].

Computing the expected steps: Since we can compute the expected time, we also can compute the number of expected steps that are needed to reach a goal state $s' \in G$ from a state $s \in S$. We can compute the expected steps on the embedded IPC. Because in IPCs there is no exit rate, we substitute the exit rate by 1, which describes one step in the IPC. For all states in the embedded IPC, the expected number of steps will be calculated as follows:

$$expSteps(s) = 0, \quad if\ s \in G,$$
$$expSteps(s) = 1 + \sum_{s' \in S} P(s, s') \cdot expSteps(s'), \quad if\ s \notin G.$$

References

[1] L. Zhang and M. Neuhäußer. Model Checking Interactive Markov Chains. *Lecture Notes in Computer Science*, 6015:53–68, 2010.