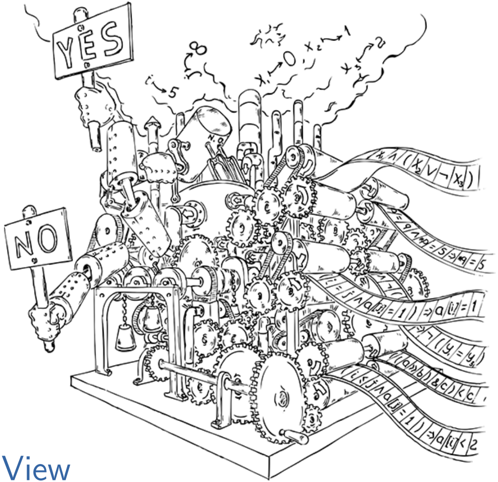
The Omega Test

Chapter 5



Decision Procedures

An Algorithmic Point of View

D.Kroening O.Strichman

Revision 1.0

The Omega Test

- Goal: Decide satisfiability of conjunction of linear constraints over integers

$$\sum_{0 \leq i \leq n} a_i x_i \geq 0$$

- Original application:
  program optimizations done by a compiler
- Extension of *Fourier-Motzkin* variable elimination:
  - Pick one variable and eliminate it
  - Continue until all variables but one are eliminated

Preprocessing (1)

Normalize coefficients: divide by the GCD

$$8x + 6y \leq 0 \quad \longrightarrow$$

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$$4y \geq 1 \quad \longrightarrow$$

Preprocessing (1)

Normalize coefficients: divide by the GCD

$$\begin{array}{lcl} 8x + 6y \leq 0 & \longrightarrow & 4x + 3y \leq 0 \\ 4y \geq 1 & \longrightarrow & y \geq \lceil 1/4 \rceil \end{array}$$

Preprocessing (1)

Normalize coefficients: divide by the GCD

$$8x + 6y \leq 0 \quad \longrightarrow \quad 4x + 3y \leq 0$$
$$4y \geq 1 \quad \longrightarrow \quad y \geq \lceil 1/4 \rceil$$

'Tightening'

Preprocessing (1)

Normalize coefficients: divide by the GCD

'Tightening'

$$
\begin{array}{lll}
8x + 6y \leq 0 & \longrightarrow & 4x + 3y \leq 0 \\
4y \geq 1 & \longrightarrow & y \geq \lceil 1/4 \rceil \\
3x + 3y = 2 & \longrightarrow &
\end{array}
$$

Normalize coefficients: divide by the GCD

'Tightening'

$$
\begin{array}{lcl}
8x + 6y \leq 0 & \longrightarrow & 4x + 3y \leq 0 \\
4y \geq 1 & \longrightarrow & y \geq \lceil 1/4 \rceil \\
3x + 3y = 2 & \longrightarrow & x + y = 2/3
\end{array}
$$

Normalize coefficients: divide by the GCD

'Tightening'

$$
\begin{array}{lll}
8x + 6y \leq 0 & \longrightarrow & 4x + 3y \leq 0 \\
4y \geq 1 & \longrightarrow & y \geq \lceil 1/4 \rceil \\
3x + 3y = 2 & \longrightarrow & x + y = 2/3 \quad \longrightarrow \quad \text{UNSAT}
\end{array}
$$

Preprocessing (2)

Eliminate equalities

- Let x_i denote a variable and a_i its coefficient, for $i = 1, 2, \ldots$
- Use *substitution* if there is a variable with coefficient $a_i = 1$

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- Otherwise, pick variable x_k from an equality, and make a_k positive

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- Use *substitution* if there is a variable with coefficient $a_i = 1$
- Otherwise, pick variable x_k from an equality, and make a_k positive

- Define $a \,\widehat{\bmod}\, b := a - b\lfloor a/b + 1/2 \rfloor$
- Let $m = a_k + 1$
- Note that $a_k \,\widehat{\bmod}\, m = -1$

Eliminate equalities

- Create new variable σ and add:

$$\sum_i (a_i \,\widehat{\bmod}\, m) x_i = m\sigma + b \,\widehat{\bmod}\, m$$

- Solve for x_k:

$$x_k = -m\sigma - b \,\widehat{\bmod}\, m + \sum_{i \neq k} a_k (a_i \,\widehat{\bmod}\, m) x_i$$

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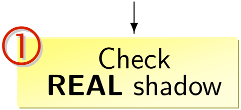
- Solve for x_k:

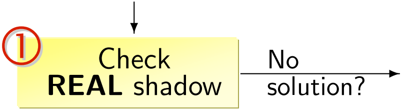
$$x_k = -m\sigma - b \,\widehat{\bmod}\, m + \sum_{i \neq k} a_k (a_i \,\widehat{\bmod}\, m) x_i$$

- Q: What is the point of adding a constraint to eliminate one?

Overview of the Omega Test



Overview of the Omega Test



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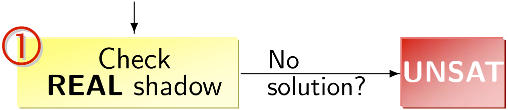
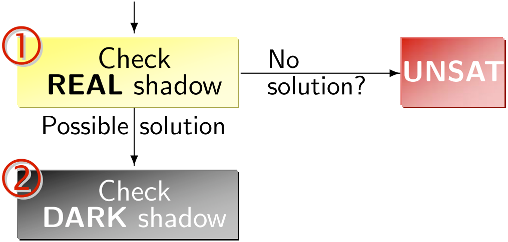


Overview of the Omega Test



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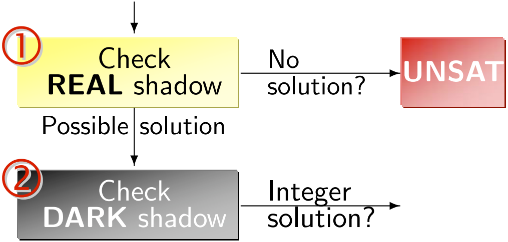
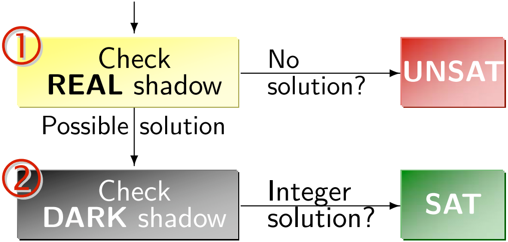
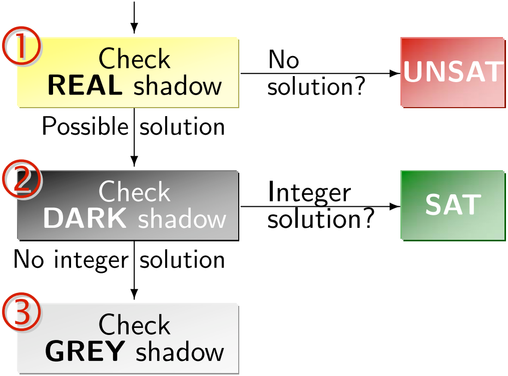



①
Check
REAL shadow
No
solution?
UNSAT
Possible solution
②
Check
DARK shadow
Integer
solution?
SAT

Overview of the Omega Test



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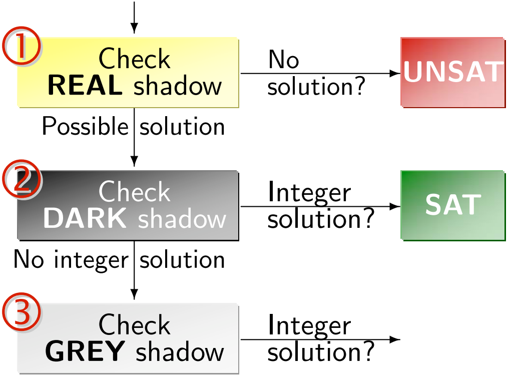
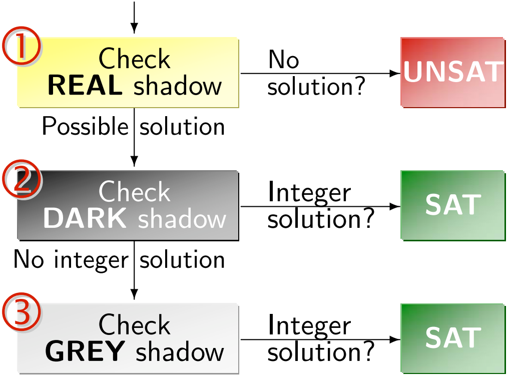
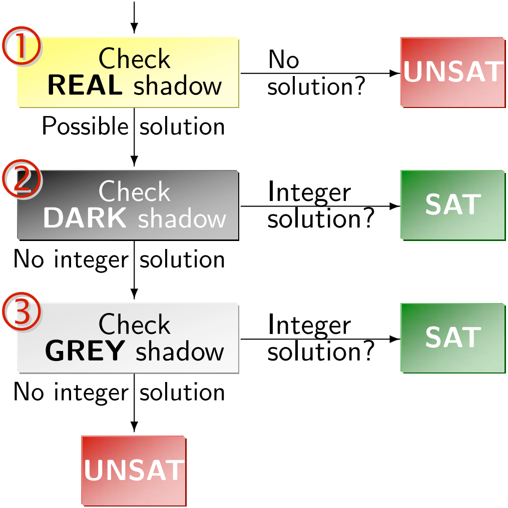



1
Check
REAL shadow
No
solution?
UNSAT
Possible solution
2
Check
DARK shadow
Integer
solution?
SAT
No integer solution
3
Check
GREY shadow
Integer
solution?
SAT

Overview of the Omega Test



① Check **REAL** shadow

- Assume we eliminate variable z
- For each pair of upper/lower bound:

$$\beta \leq bz \qquad cz \leq \gamma$$

The real shadow

① Check **REAL** shadow

- Assume we eliminate variable z
- For each pair of upper/lower bound:

$$\begin{array}{rclcrcl} \beta & \leq & bz & \quad & cz & \leq & \gamma \\ c\beta & \leq & cbz & \quad & cbz & \leq & b\gamma \end{array}$$

The real shadow

① Check **REAL** shadow

- Assume we eliminate variable z
- For each pair of upper/lower bound:

$$\begin{array}{rcl} \beta & \leq & bz \\ c\beta & \leq & cbz \end{array} \qquad \begin{array}{rcl} cz & \leq & \gamma \\ cbz & \leq & b\gamma \end{array}$$

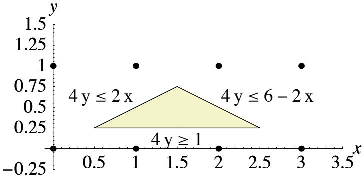
- Constraint for real shadow:

$$c\beta \leq b\gamma$$

- Add this constraint, and call Omega recursively!

The real shadow: Example I

① Check **REAL** shadow

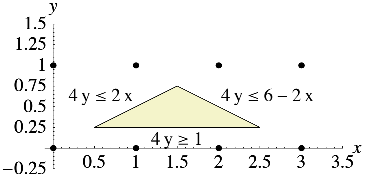


$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

1 Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

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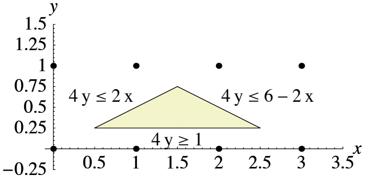
Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x + 6$

The real shadow: Example I

① Check **REAL** shadow



$4y \leq 2x$

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$4y \geq 1$

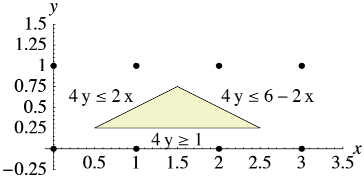
Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x + 6$

$$4y \leq 6 - 4y$$

The real shadow: Example I

① Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

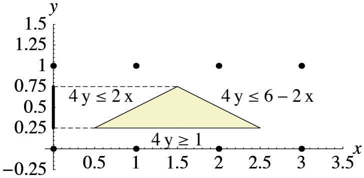
$4y \geq 1$

Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x + 6$

$$4y \leq 6 - 4y$$
$$8y \leq 6$$

① Check **REAL** shadow



$4y \;\leq\; 2x$

$4y \;\leq\; -2x+6$

$4y \;\geq\; 1$

Eliminate x:

$4y \;\leq\; 2x$ $\quad$ $4y \;\leq\; -2x+6$

$$4y \leq 6 - 4y$$
$$8y \leq 6$$

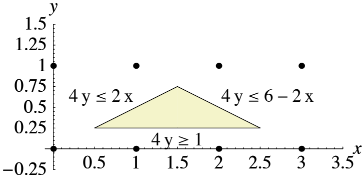
Real Shadow:

$8y \leq 6$, $4y \geq 1$ ⟹ $y \leq 0.75$, $y \geq 0.25$

No integer solution

$\Longrightarrow$ Original problem has no solution

1 Check
REAL shadow



$4y \leq 2x$

$4y \leq -2x + 6$

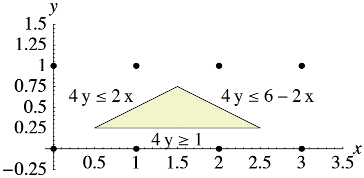
$4y \geq 1$

Let's eliminate y instead:

$1 \leq 4y$ $\quad$ $4y \leq 2x$

The real shadow: Example II

① Check **REAL** shadow



$4y \le 2x$

$4y \le -2x + 6$

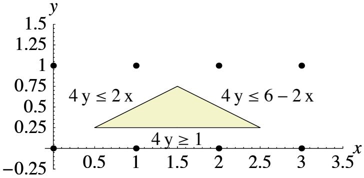
$4y \ge 1$

Let's eliminate y instead:

$1 \le 4y$ $\quad$ $4y \le 2x$

$1 \le 2x$

1 Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

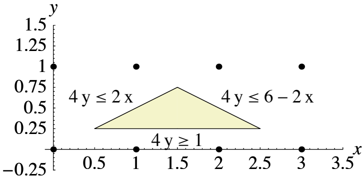
Let's eliminate y instead:

$1 \leq 4y$ $\quad$ $4y \leq 2x$

$1 \leq 2x$

$1 \leq 4y$ $\quad$ $4y \leq -2x + 6$

1 Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

Let's eliminate y instead:

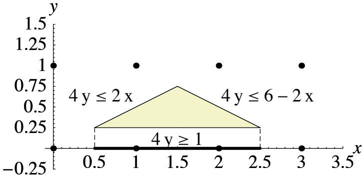
$1 \leq 4y$ $4y \leq 2x$

$1 \leq 2x$

$1 \leq 4y$ $4y \leq -2x + 6$

$1 \leq -2x + 6$

① Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

Let's eliminate y instead:

$1 \leq 4y$ and $4y \leq 2x$ → $1 \leq 2x$

$1 \leq 4y$ and $4y \leq -2x + 6$ → $1 \leq -2x + 6$

Real Shadow:

$1 \leq 2x$

$1 \leq -2x + 6$

⇒

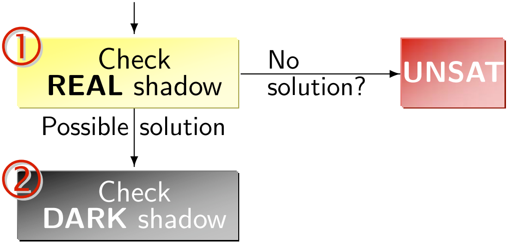
$x \leq 0.5$

$x \geq 2.5$

Integer solution!

But original problem has no integer solution!

From Real to Dark Shadow



- An integer solution for the REAL shadow does not guarantee that there is an integer solution for the original problem
- Thus, we check the DARK shadow next

Idea of the dark shadow

② **Check DARK shadow**

- Idea of the DARK shadow:

$$\beta \leq bz \qquad\qquad cz \leq \gamma$$

Idea of the dark shadow

② Check **DARK** shadow

- Idea of the DARK shadow:

$$\begin{array}{rclr} \beta & \leq & bz & \mid : b \\ \frac{\beta}{b} & \leq & z & \end{array} \qquad \begin{array}{rclr} cz & \leq & \gamma & \mid : c \\ z & \leq & \frac{\gamma}{c} & \end{array} \qquad z \in \mathbb{N}$$

② Check **DARK** shadow

- Idea of the DARK shadow:

$$\begin{array}{rclcrclcl} \beta & \leq & bz & |:b & cz & \leq & \gamma & |:c & \\ \frac{\beta}{b} & \leq & z & & z & \leq & \frac{\gamma}{c} & & z \in \mathbb{N} \end{array}$$

- How to compute the dark shadow?
- Try to *prove* that there is an integer z between $\frac{\beta}{b}$ and $\frac{\gamma}{c}$

Dark shadow: Proof by contradiction

② Check **DARK** shadow

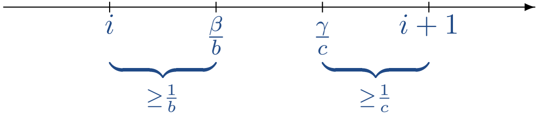
Assume there is no integer z between $\frac{\beta}{b}$ and $\frac{\gamma}{c}$.

Dark shadow: Proof by contradiction

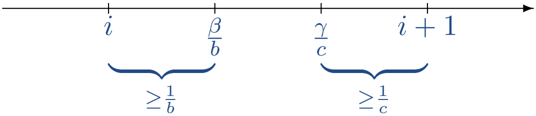
② Check **DARK** shadow

Assume there is no integer z between $\frac{\beta}{b}$ and $\frac{\gamma}{c}$. Then:

Let $i := \lfloor \frac{\beta}{b} \rfloor \qquad i \in \mathbb{Z}$

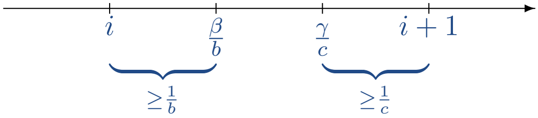


Dark shadow: Proof by contradiction

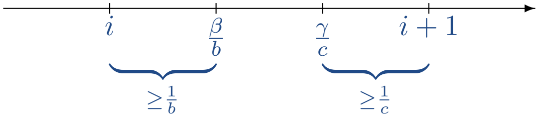


$$\begin{aligned} \frac{\beta}{b} - i &\geq \frac{1}{b} \\ i + 1 - \frac{\gamma}{c} &\geq \frac{1}{c} \end{aligned}$$

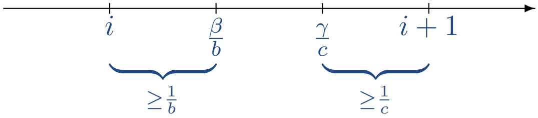
Dark shadow: Proof by contradiction



$$\begin{array}{rcl} \frac{\beta}{b} - i & \geq & \frac{1}{b} \\ i + 1 - \frac{\gamma}{c} & \geq & \frac{1}{c} \\ \hline \frac{\beta}{b} + 1 - \frac{\gamma}{c} & \geq & \frac{1}{b} + \frac{1}{c} \end{array}$$

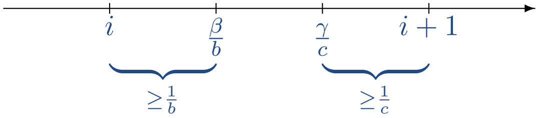


$$\begin{array}{rcl} \frac{\beta}{b} - i & \geq & \frac{1}{b} \\ i + 1 - \frac{\gamma}{c} & \geq & \frac{1}{c} \\ \hline \frac{\beta}{b} + 1 - \frac{\gamma}{c} & \geq & \frac{1}{b} + \frac{1}{c} \qquad | \cdot c \cdot b \\ c\beta + cb - b\gamma & \geq & c + b \end{array}$$



$$
\begin{array}{rcll}
\frac{\beta}{b} - i & \geq & \frac{1}{b} & \\
i + 1 - \frac{\gamma}{c} & \geq & \frac{1}{c} & \\
\hline
\frac{\beta}{b} + 1 - \frac{\gamma}{c} & \geq & \frac{1}{b} + \frac{1}{c} & \quad | \cdot c \cdot b \\
c\beta + cb - b\gamma & \geq & c + b & \quad | - cb \\
c\beta - b\gamma & \geq & -cb + c + b &
\end{array}
$$

Dark shadow: Proof by contradiction



$$
\begin{array}{rcll}
\frac{\beta}{b} - i & \geq & \frac{1}{b} & \\
i + 1 - \frac{\gamma}{c} & \geq & \frac{1}{c} & \\
\hline\hline
\frac{\beta}{b} + 1 - \frac{\gamma}{c} & \geq & \frac{1}{b} + \frac{1}{c} & | \cdot c \cdot b \\
c\beta + cb - b\gamma & \geq & c + b & | - cb \\
c\beta - b\gamma & \geq & -cb + c + b & | \cdot (-1) \\
b\gamma - c\beta & \leq & cb - c - b &
\end{array}
$$

Dark shadow: Proof by contradiction

- From previous slide:

$$b\gamma - c\beta \quad \leq \quad cb - c - b$$

Dark shadow: Proof by contradiction

- From previous slide:

$$\begin{array}{rrcl} & b\gamma - c\beta & \leq & cb - c - b \\ \iff & \neg(b\gamma - c\beta & > & cb - c - b) \end{array}$$

Dark shadow: Proof by contradiction

- From previous slide:

$$
\begin{array}{rrcl}
 & b\gamma - c\beta & \leq & cb - c - b \\
\iff & \neg(b\gamma - c\beta & > & cb - c - b) \\
\iff & \neg(b\gamma - c\beta & \geq & cb - c - b + 1)
\end{array}
$$

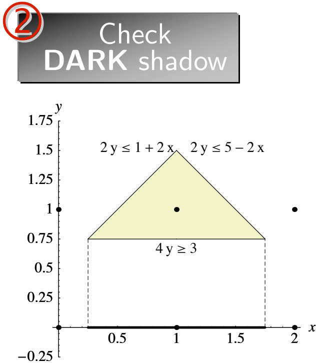
Dark shadow: Proof by contradiction

- From previous slide:

$$\begin{array}{rrcl} & b\gamma - c\beta & \leq & cb - c - b \\ \iff & \neg(b\gamma - c\beta & > & cb - c - b) \\ \iff & \neg(b\gamma - c\beta & \geq & cb - c - b + 1) \\ \iff & \neg(\underbrace{b\gamma - c\beta \; \geq \; (c-1)(b-1)}_{*}) \end{array}$$

- Thus, if * holds, we know that there must be an integer solution.
- If $c = 1$ or $b = 1$, then this is the same as the real shadow. This case is called an exact projection.

Example for the dark shadow

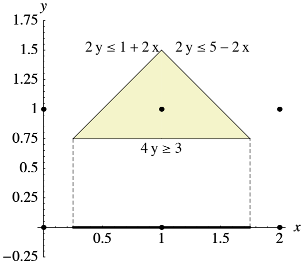


$$2y \leq 2x + 1$$

$$2y \leq -2x + 5$$

$$4y \geq 3$$

② Check **DARK** shadow



$$2y \le 2x+1$$

$$2y \le -2x+5$$

$$4y \ge 3$$

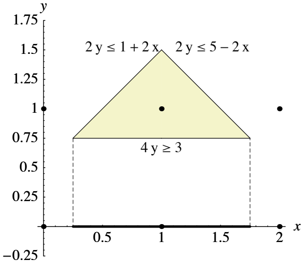
Eliminate y with the dark shadow:

$$2y \le 2x+1 \qquad 4y \ge 3$$

$$4(2x+1) - 2\cdot 3 \ge (2-1)(4-1)$$

Example for the dark shadow

② Check **DARK** shadow



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$2y \leq -2x + 5$

$4y \geq 3$

Eliminate y with the dark shadow:

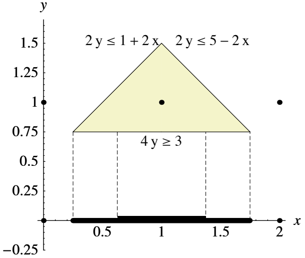
$2y \leq 2x + 1$ $\quad$ $4y \geq 3$

$$4(2x + 1) - 2 \cdot 3 \geq (2 - 1)(4 - 1)$$

$4y \geq 3$ $\quad$ $2y \leq -2x + 5$

$$4(-2x + 5) - 2 \cdot 3 \geq (2 - 1)(4 - 1)$$

② Check **DARK** shadow



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$$2y \leq -2x + 5$$

$$4y \geq 3$$

Eliminate y with the dark shadow:

$2y \leq 2x + 1$ $\quad$ $4y \geq 3$

$$4(2x + 1) - 2 \cdot 3 \geq (2 - 1)(4 - 1)$$

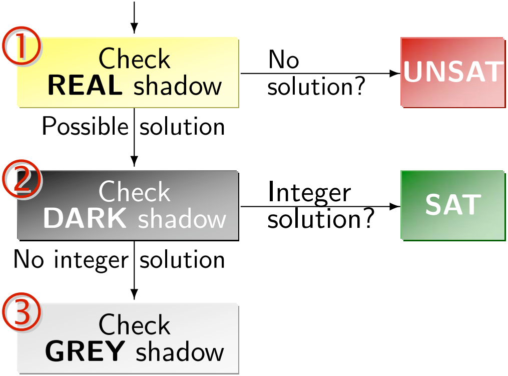
$4y \geq 3$ $\quad$ $2y \leq -2x + 5$

$$4(-2x + 5) - 2 \cdot 3 \geq (2 - 1)(4 - 1)$$

Dark Shadow:

$$\begin{array}{lll} x & \geq & 5/8 \\ x & \leq & 11/8 \end{array}$$

$\Longrightarrow$ Integer solution!



- No integer solution in the DARK shadow does not guarantee that there is no integer solution for the original problem
- Thus, we check the GREY shadow next

③ Check **GREY** shadow

Idea of the Grey shadow

If the real shadow R has integer solutions,
but the dark shadow D does not, search $R \setminus D$.

The grey shadow

③ Check **GREY** shadow

Idea of the Grey shadow

If the real shadow R has integer solutions,
but the dark shadow D does not, search $R \setminus D$.

$$
\begin{array}{rl}
\text{In R:} & b\gamma \geq cbz \geq c\beta \\
\textbf{Not}\text{ in D:} & cb - c - b \geq b\gamma - c\beta \\
\iff & cb - c - b + c\beta \geq b\gamma \\
\Rightarrow & cb - c - b + c\beta \geq cbz \geq c\beta
\end{array}
$$

③ Check **GREY** shadow

Idea of the Grey shadow

If the real shadow R has integer solutions,
but the dark shadow D does not, search $R \setminus D$.

$$\begin{array}{rl}
\text{In R:} & b\gamma \geq cbz \geq c\beta \\
\textbf{Not}\text{ in D:} & cb - c - b \geq b\gamma - c\beta \\
\iff & cb - c - b + c\beta \geq b\gamma \\
\Rightarrow & cb - c - b + c\beta \geq cbz \geq c\beta \qquad | : c \\
 & (cb - c - b)/c + \beta \geq bz \geq \beta
\end{array}$$

③ Check **GREY** shadow

- Try all values of z such that

$$(cb - c - b)/c + \beta \geq bz \geq \beta$$

③ Check **GREY** shadow

- Try all values of z such that

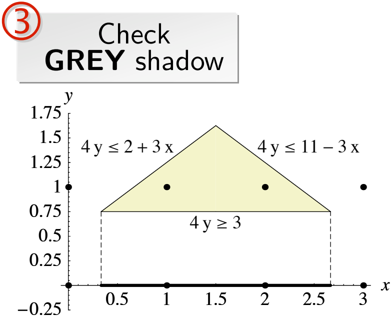
$$(cb - c - b)/c + \beta \geq bz \geq \beta$$

- Optimization: find the largest coefficient c in any upper bound and try the following for each lower bound $bz \geq \beta$:

$$bz = \beta + i \quad \text{for } (cb - c - b)/c \geq i \geq 0$$

- As before, combine this with the original problem, and solve recursively.

Example of the grey shadow



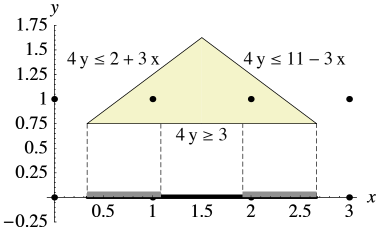
$$4y \leq 3x + 2$$

$$4y \leq -3x + 11$$

$$4y \geq 3$$

Example of the grey shadow

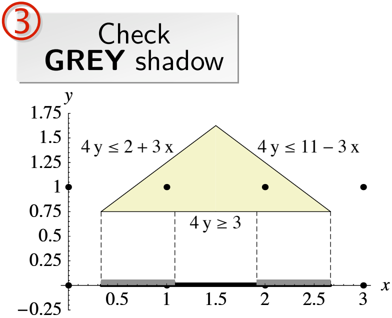
③ Check **GREY** shadow



$$4y \leq 3x + 2$$

$$4y \leq -3x + 11$$

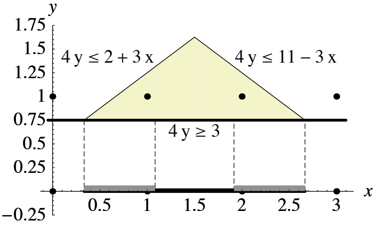
$$4y \geq 3$$



$$4y \le 3x + 2$$
$$4y \le -3x + 11$$
$$4y \ge 3$$

- Eliminate y:
  $c = 4$, $b = 4$, $\beta = 3$
- New constraint:
  $4y = 3 + i$ for
  $2 \ge i \ge 0$:

③ Check **GREY** shadow



$$4y \leq 3x + 2$$

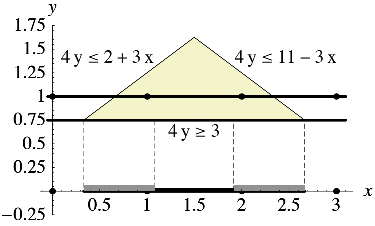
$$4y \leq -3x + 11$$

$$4y \geq 3$$

- Eliminate y:
  $c = 4$, $b = 4$, $\beta = 3$
- New constraint:
  $4y = 3 + i$ for
  $2 \geq i \geq 0$:

  $$4y = 3$$

③ Check **GREY** shadow



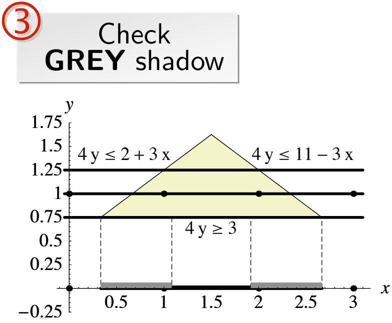
$$4y \leq 3x + 2$$

$$4y \leq -3x + 11$$

$$4y \geq 3$$

- Eliminate y:
  $c = 4$, $b = 4$, $\beta = 3$
- New constraint:
  $4y = 3 + i$ for
  $2 \geq i \geq 0$:
  $$4y = 3$$
  $$4y = 4$$

Example of the grey shadow



$$4y \leq 3x + 2$$

$$4y \leq -3x + 11$$

$$4y \geq 3$$

- Eliminate y:
  $c = 4$, $b = 4$, $\beta = 3$
- New constraint:
  $4y = 3 + i$ for
  $2 \geq i \geq 0$:

  $$4y = 3$$
  $$4y = 4$$
  $$4y = 5$$

$\Longrightarrow$ Integer solution with $4y = 4$