Virtual Substitution for SMT-Solving

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Motivation

Are there solutions for x, y in the domain $\mathbb{R}$, such that:

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Real algebraic constraints:

- $p(x,\ y, \ldots) \sim 0$
- $\sim \in \{=, \neq, <, >, \leq, \geq\}$
- p multivariate polynomial in $x,\ y, \ldots$

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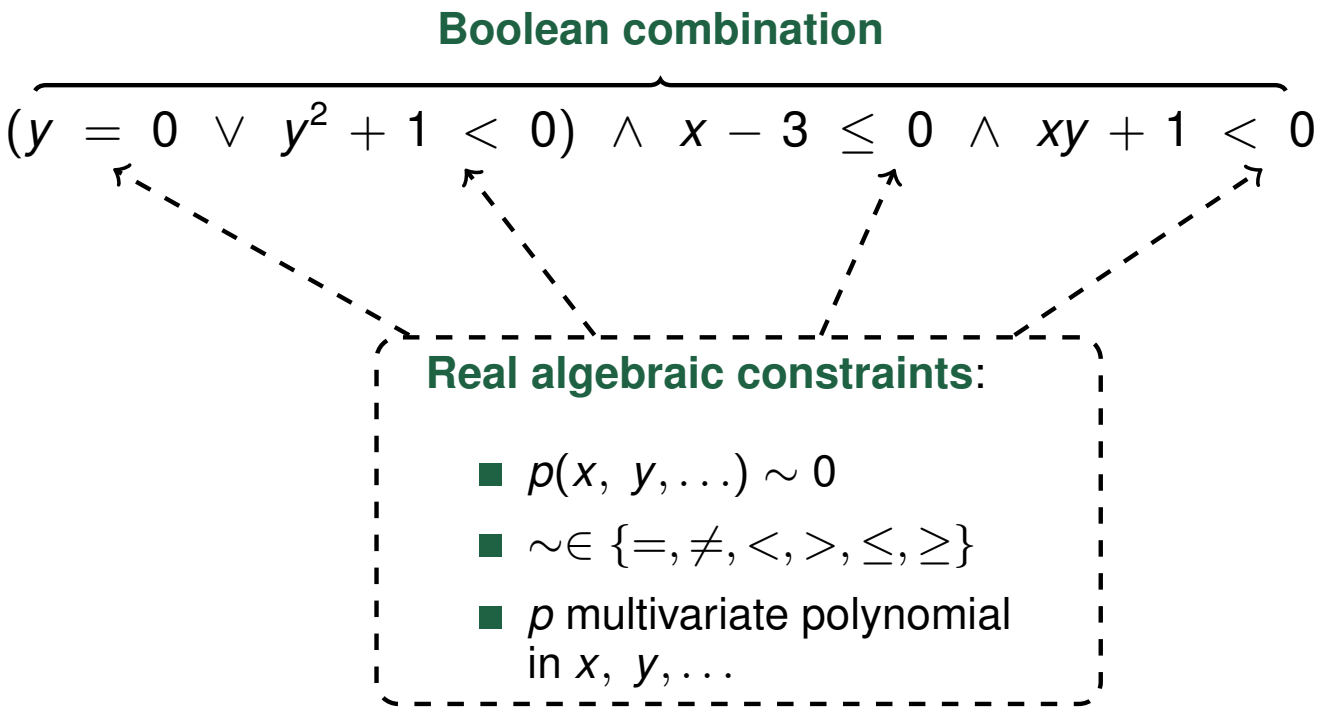

Approaches for nonlinear real algebraic formulas

Quantifier elimination:

- Methods:
  - *Cylindrical algebraic decomposition*
  - *Gröbner basis*
  - *Virtual substitution*
  - *Realizable sign conditions*
- Eliminates the variables iteratively
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- Tools: `HySAT`, `ABsolver`, `Z3`
- Benefits from a SAT-solver
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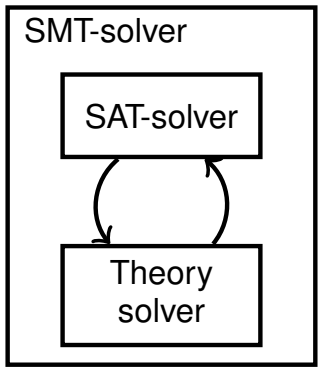

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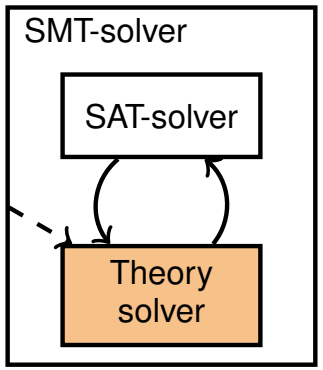

Structure of the talk

1.) Virtual substitution

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2.) Adaption to SMT
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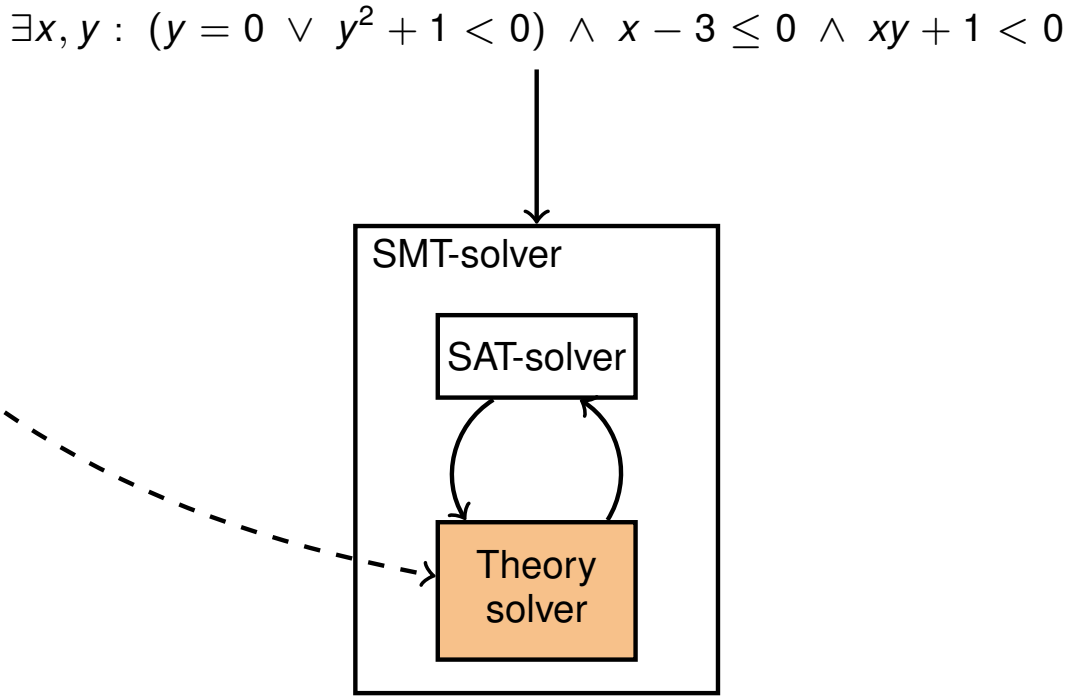

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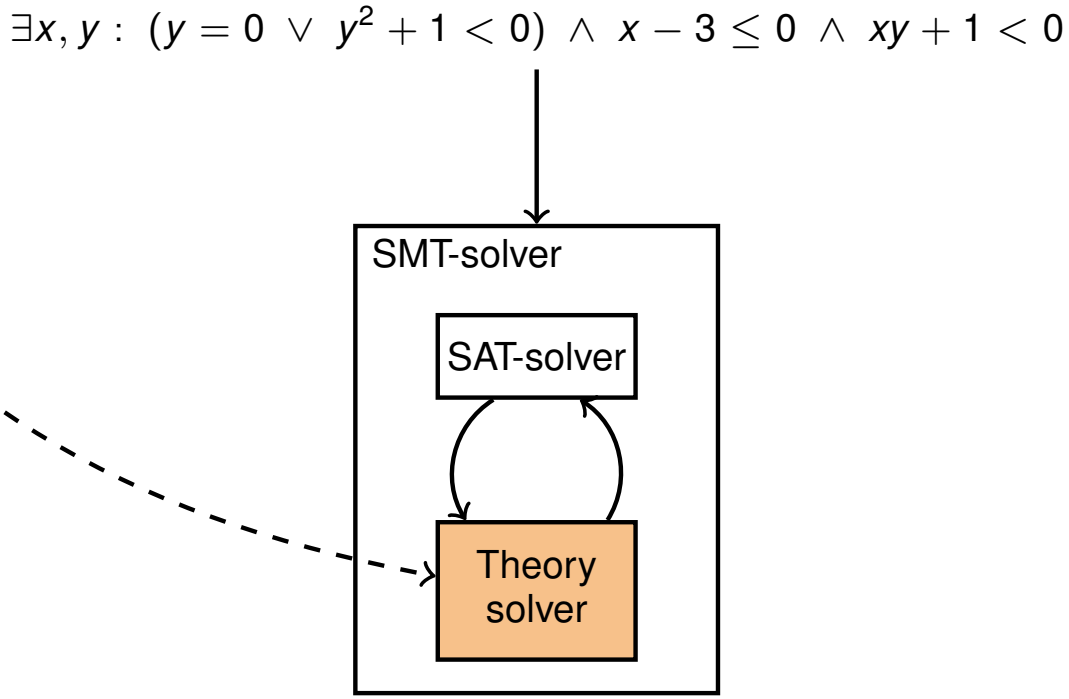
3.) Experimental results



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  1. Zeros of the polynomial p plus an infinitesimal ϵ
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- Example: $xy + 1 < 0 \quad \rightarrow \quad \begin{cases} \frac{1}{y} + \epsilon & \text{if } y \neq 0 \\ -\infty \end{cases}$

Construction of a **finite** number of solution candidates for a variable

For our example: $\exists x, y : \ (y = 0 \ \vee \ y^2 + 1 < 0) \ \wedge \ x - 3 \leq 0 \ \wedge \ xy + 1 < 0$

Test candidates

eliminate x

$\Leftrightarrow$

$$\begin{aligned}
\exists y : \quad & ((y = 0 \ \vee \ y^2 + 1 < 0) \ \wedge \ x - 3 \leq 0 \ \wedge \ xy + 1 < 0)[-\infty/\mathbf{x}] \\
\vee \quad & ((y = 0 \ \vee \ y^2 + 1 < 0) \ \wedge \ x - 3 \leq 0 \ \wedge \ xy + 1 < 0)[\mathbf{3}/\mathbf{x}] \quad) \\
\vee (\ \mathbf{y \neq 0} \quad \wedge \quad & ((y = 0 \ \vee \ y^2 + 1 < 0) \ \wedge \ x - 3 \leq 0 \ \wedge \ xy + 1 < 0)[-\tfrac{\mathbf{1}}{\mathbf{y}} + \epsilon/\mathbf{x}] \quad)
\end{aligned}$$

Side condition

eliminate y

$\Leftrightarrow$

$\cdots$

How to apply $\varphi[t/x]$

Standard substitution:

- Leads to terms containing $-\infty$, $\sqrt{\ }$ and fractions
- Introduces new variables ϵ

Example:

$$(x + 2y = 0)\ [\frac{\sqrt{4y}}{z}/x] \qquad \mapsto \qquad \frac{\sqrt{4y}}{z} + 2y = 0$$

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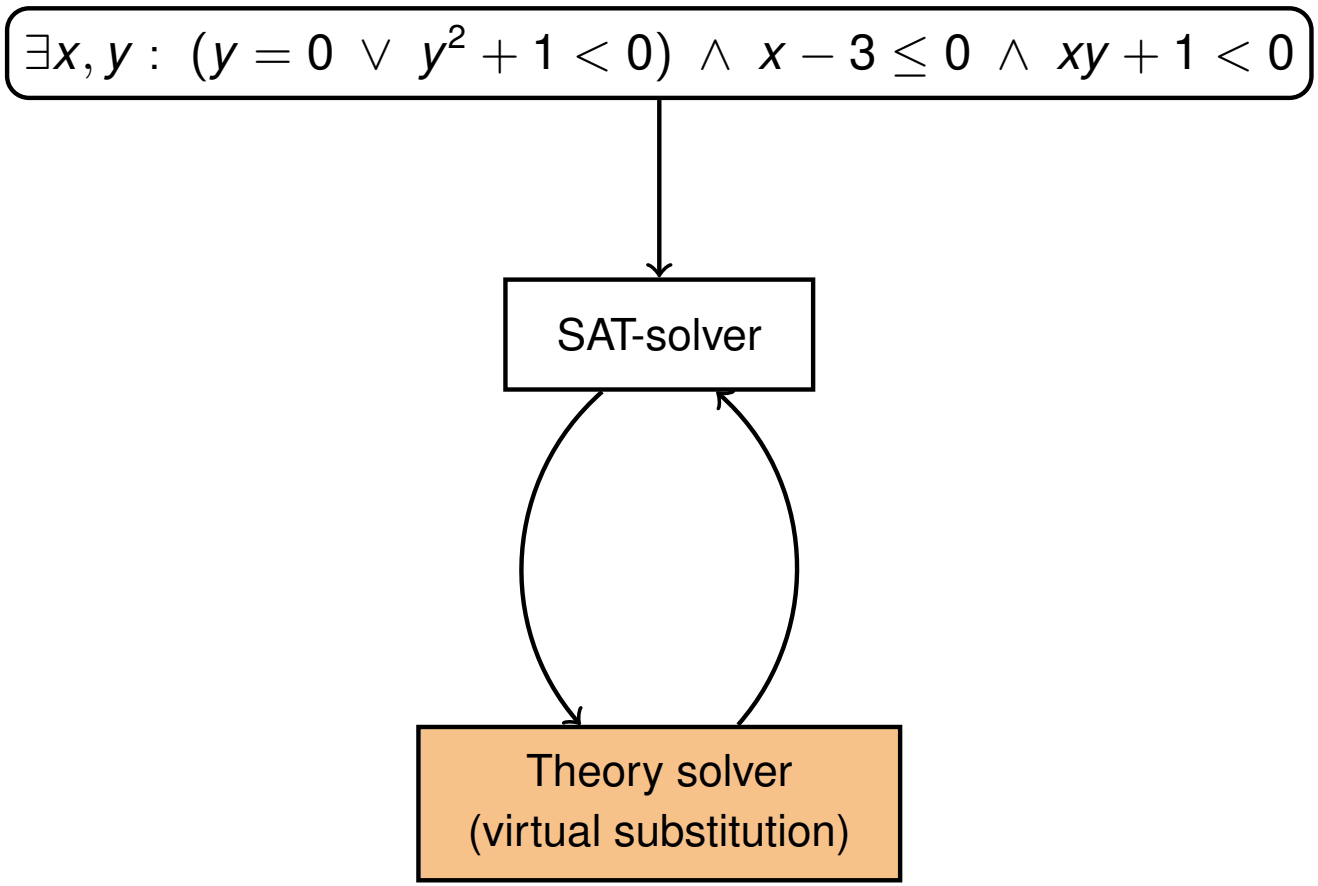
Substitution rules

Virtual substitution defines rules, which give an equivalent DNF over $(\mathbb{R}, +, \cdot, 0, 1, <)$ to the expression resulting by the above standard substitution.

Example:

$$\frac{2yz + \sqrt{4y}}{z} = 0 \quad \Leftrightarrow \quad 2yz \leq 0 \quad \wedge \quad (2yz)^2 - 4y = 0$$

SMT-solver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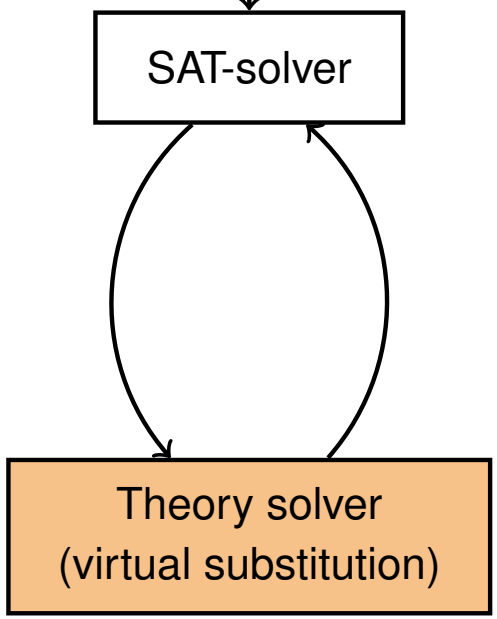
$$\exists x, y: \ (y = 0 \ \vee \ y^2 + 1 < 0) \ \wedge \ x - 3 \leq 0 \ \wedge \ xy + 1 < 0$$

Boolean abstraction: $(A \ \vee \ B) \ \wedge \ C \ \wedge \ D$



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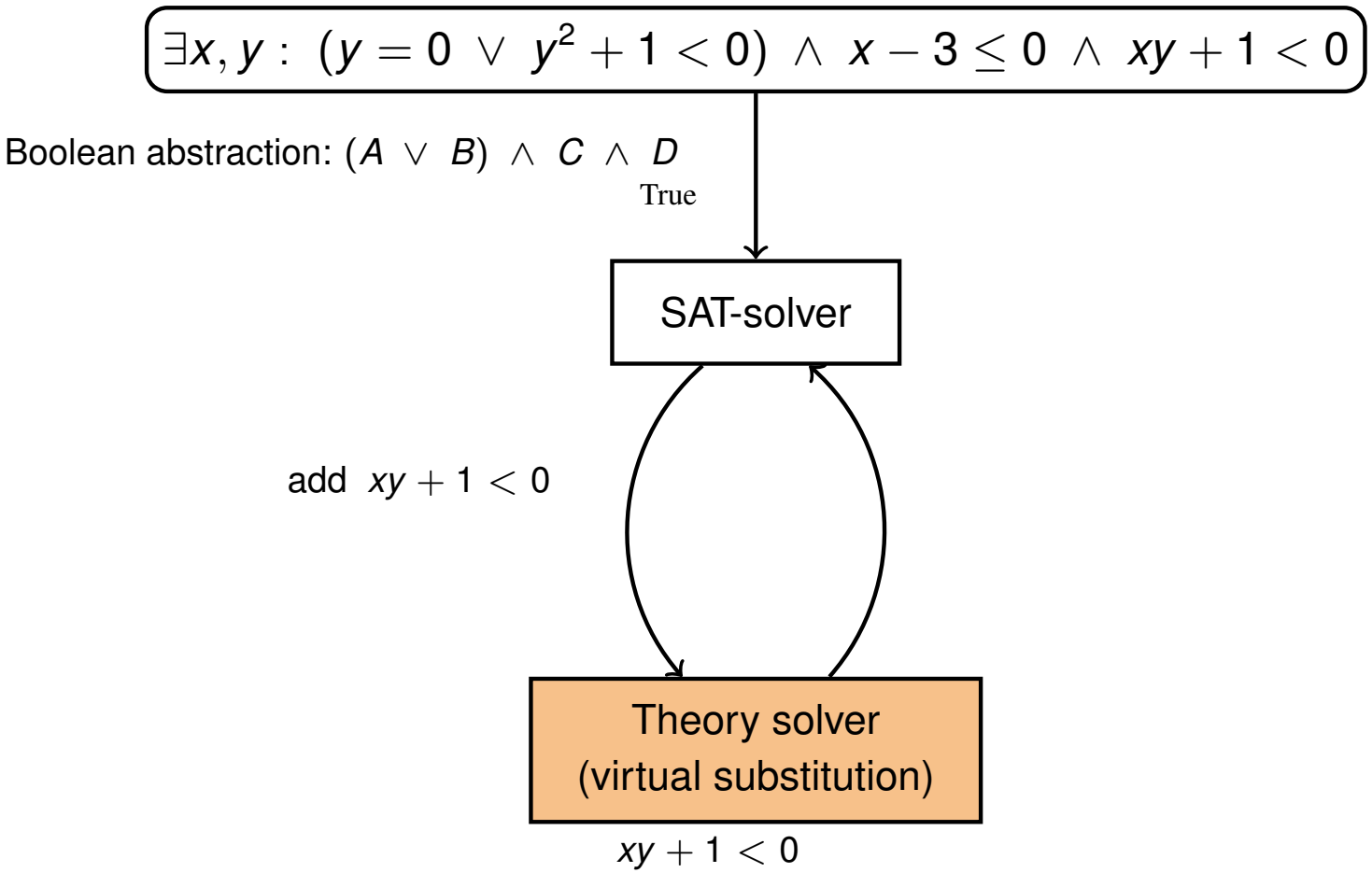
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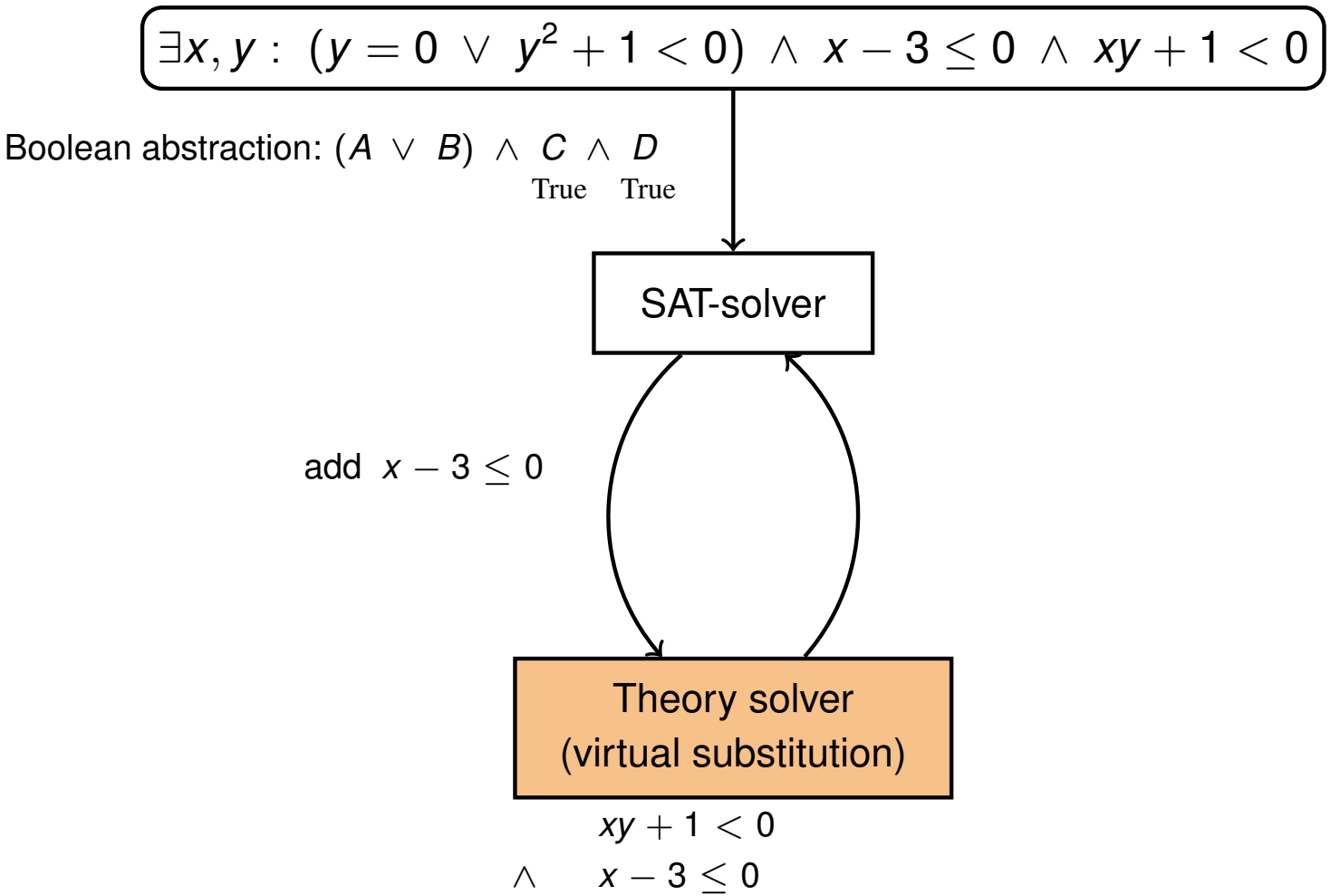
SAT-solver
add $xy + 1 < 0$
Theory solver
(virtual substitution)
$xy + 1 < 0$

SMT-solver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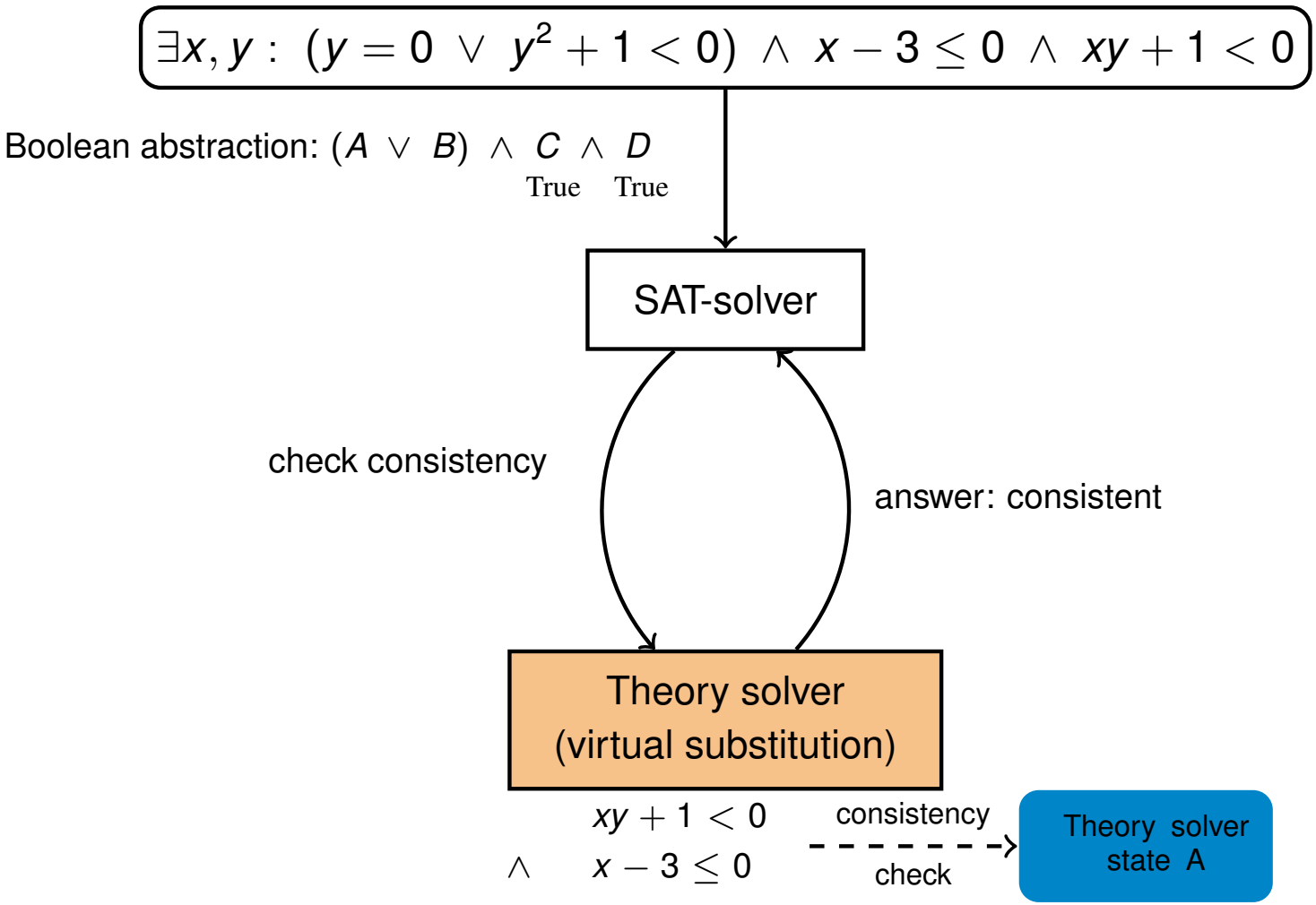
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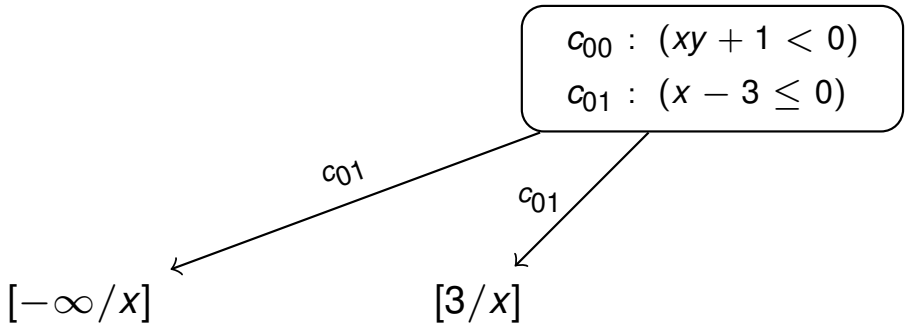
Boolean abstraction: $(A \ \vee \ B) \ \wedge \ C \ \wedge \ D$
True
True
SAT-solver
check consistency
answer: consistent
Theory solver
(virtual substitution)
$xy + 1 < 0$
$\wedge \quad x - 3 \leq 0$
consistency
check
Theory solver
state A

Theory solver: add $xy + 1 < 0$ and $x - 3 \leq 0$ and check for consistency

$$c_{00} : (xy + 1 < 0)$$
$$c_{01} : (x - 3 \leq 0)$$

Theory solver: add $xy + 1 < 0$ and $x - 3 \leq 0$ and check for consistency



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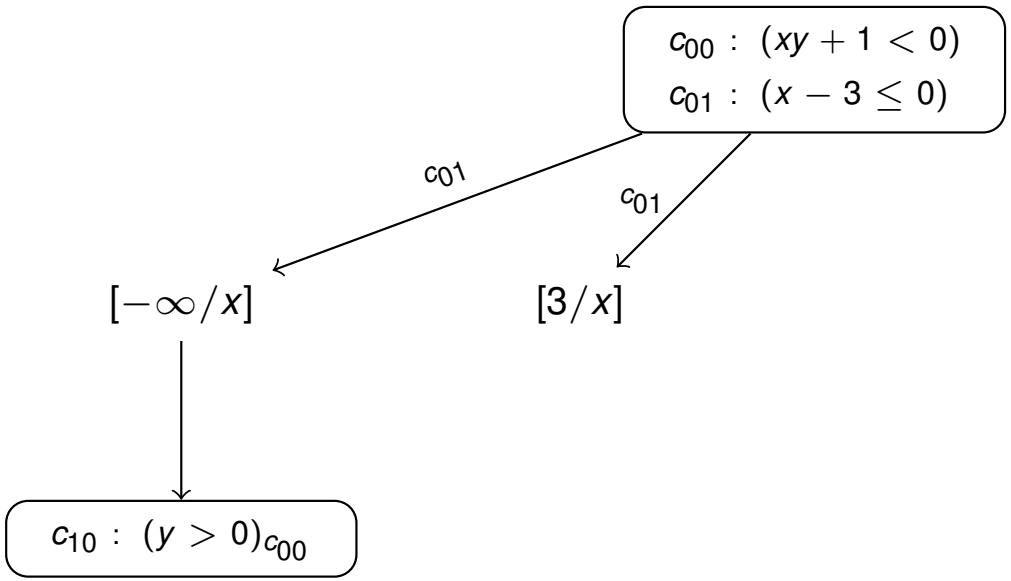

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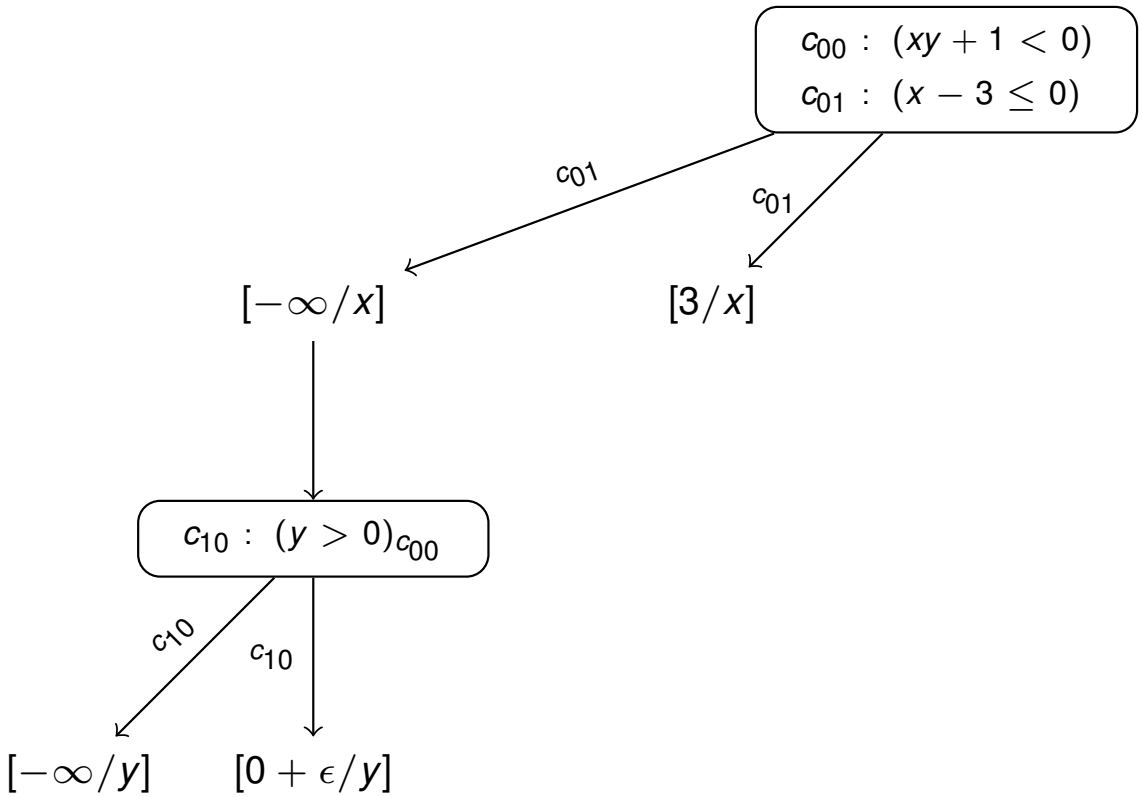

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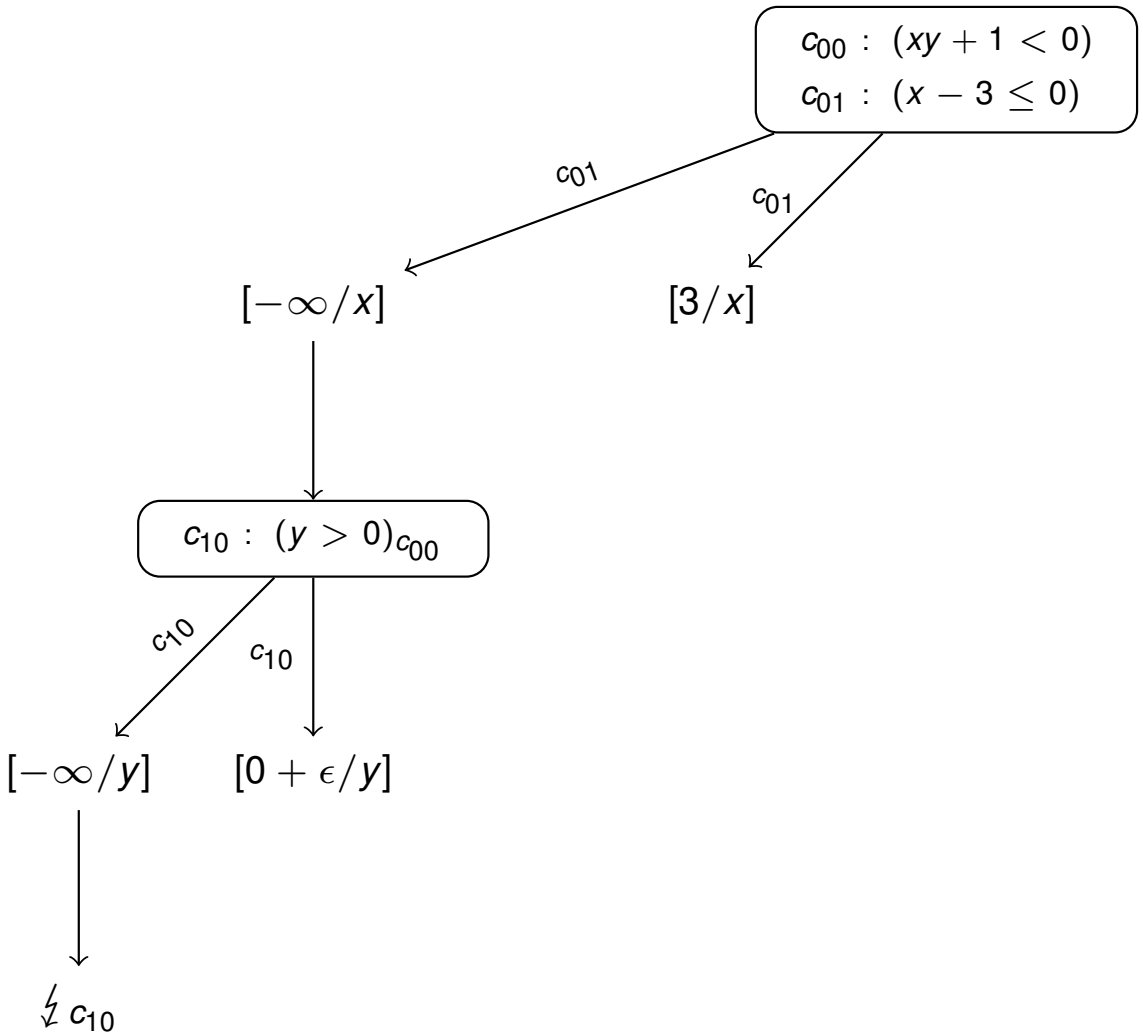

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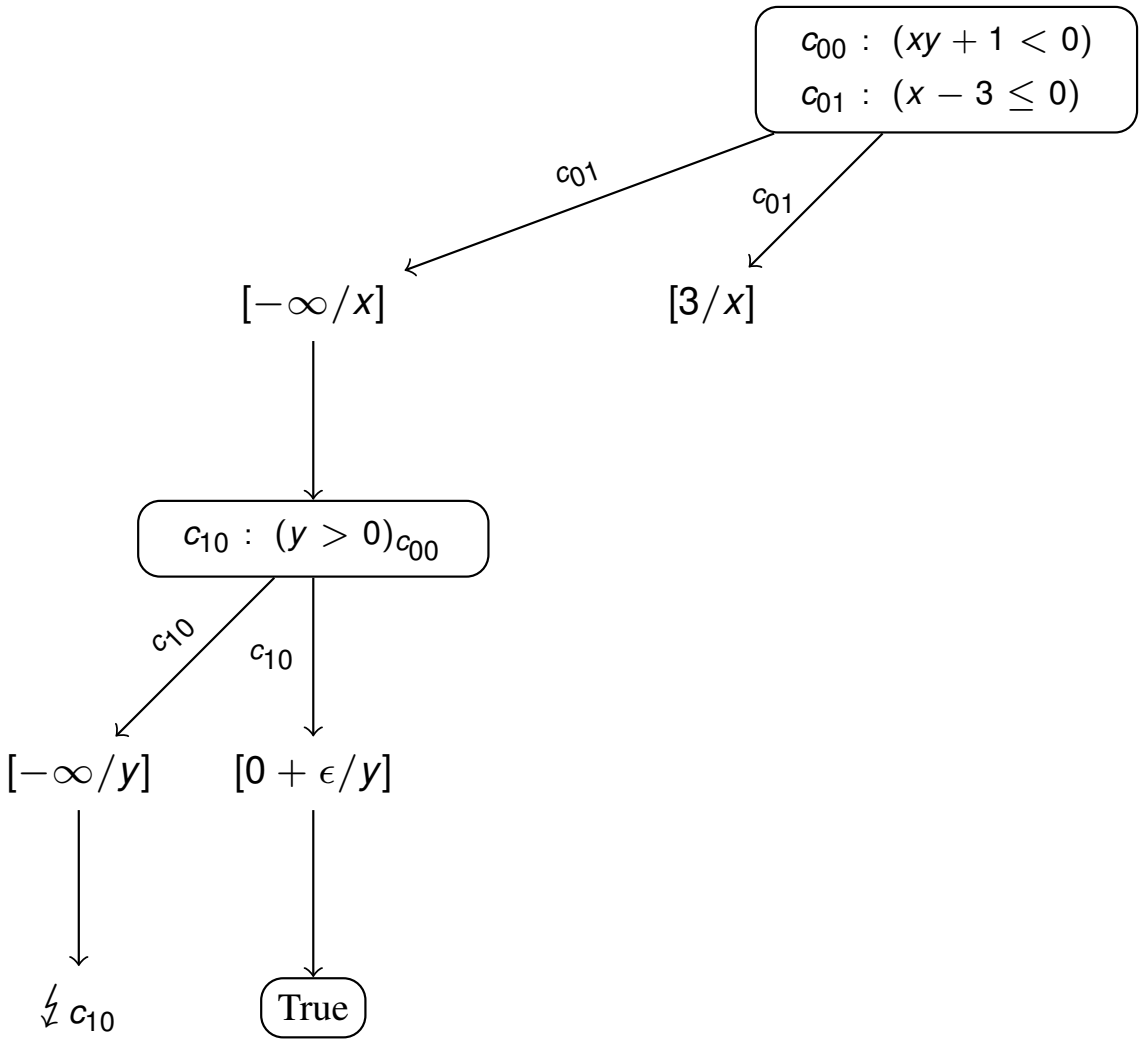

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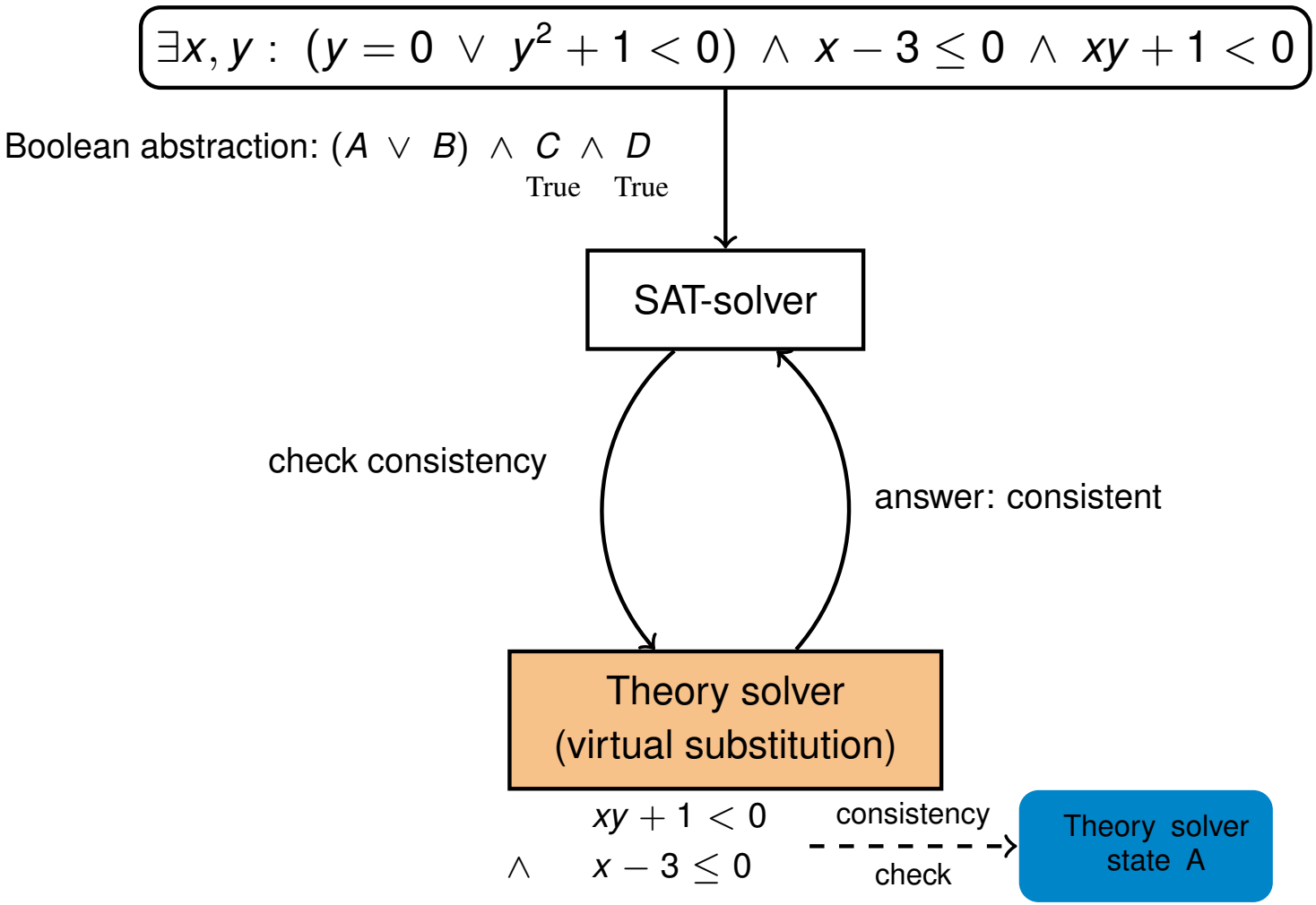
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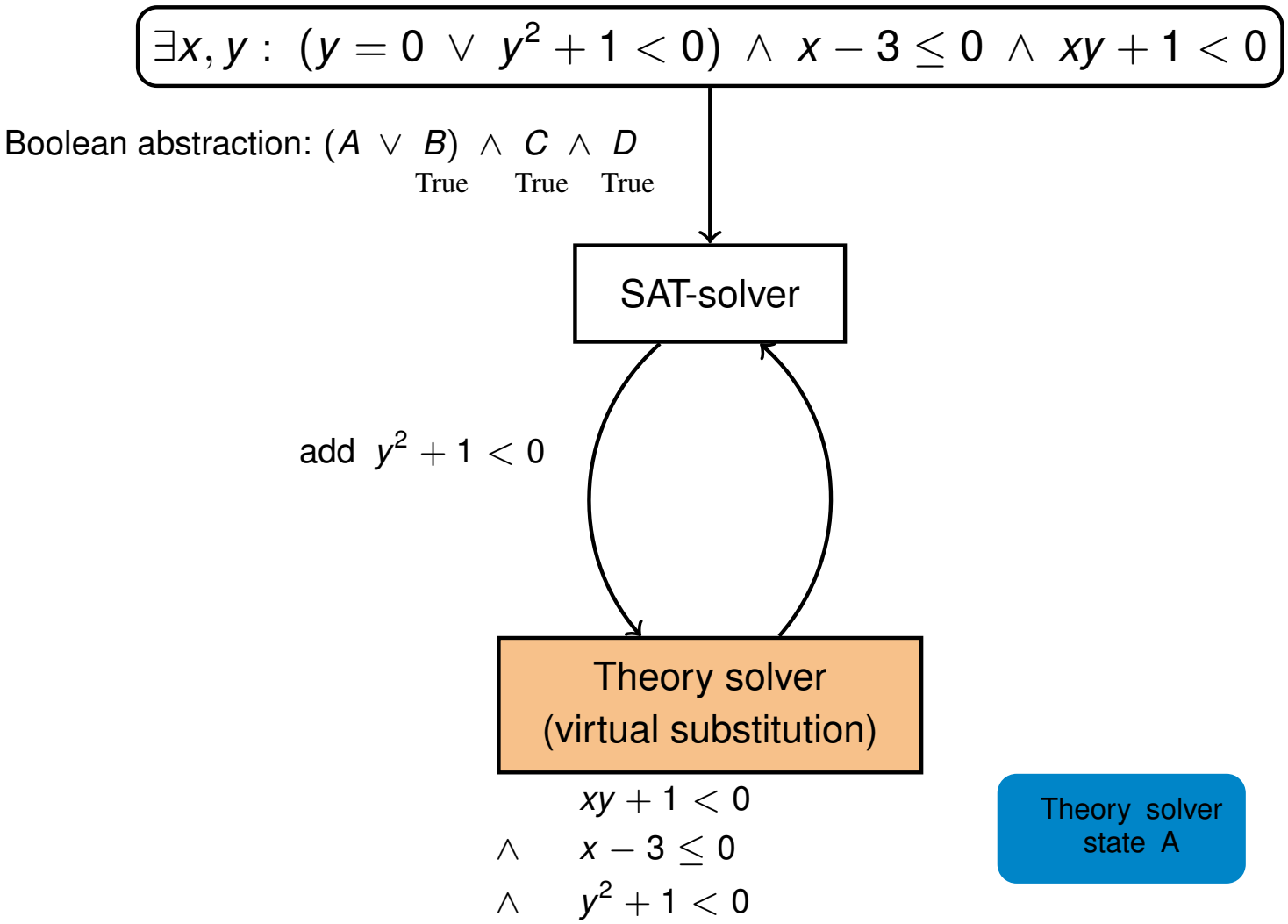
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Theory solver state A

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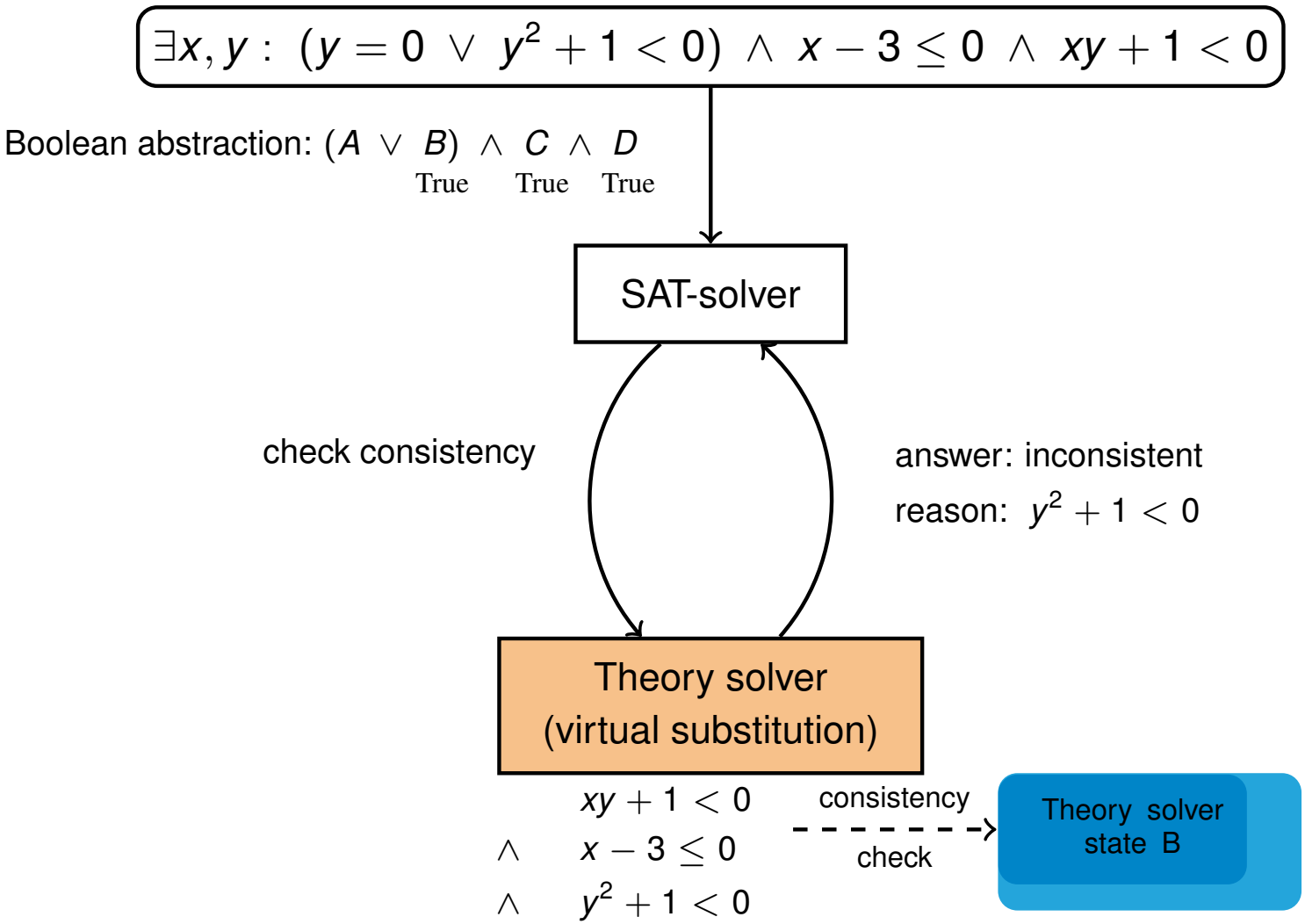
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answer: inconsistent
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Theory solver
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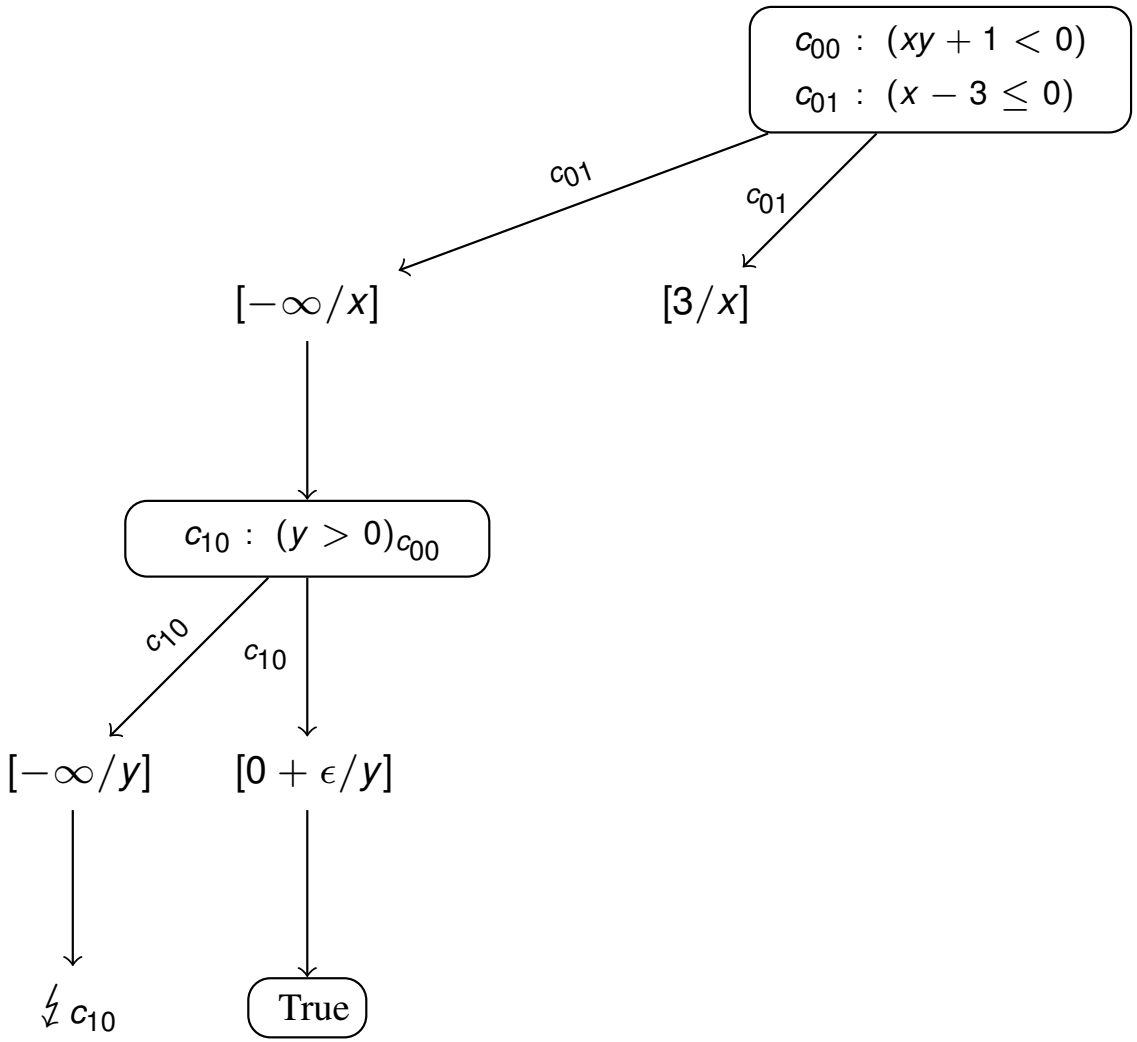
Theory solver: add $y^2 + 1 < 0$ and check for consistency



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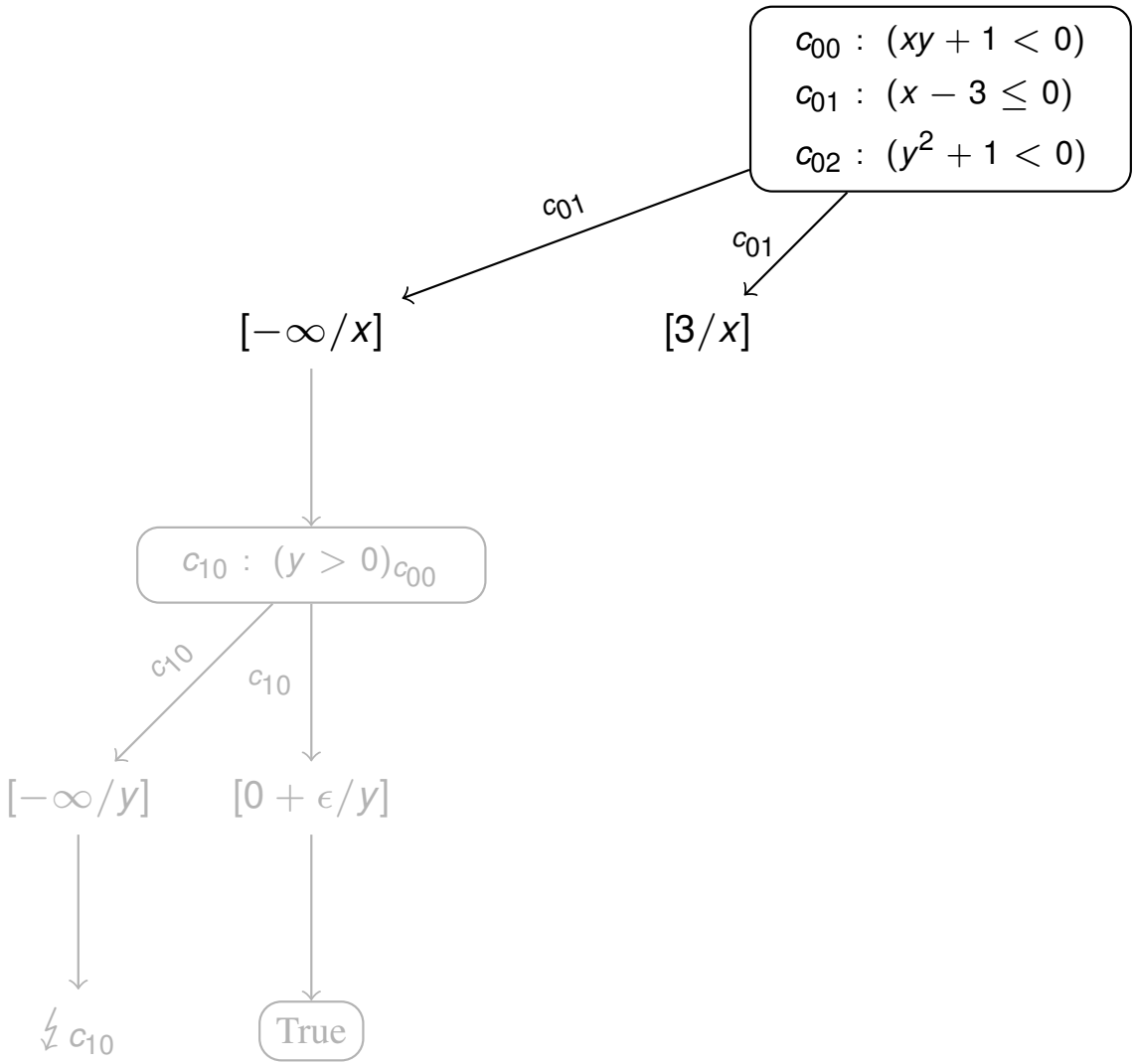

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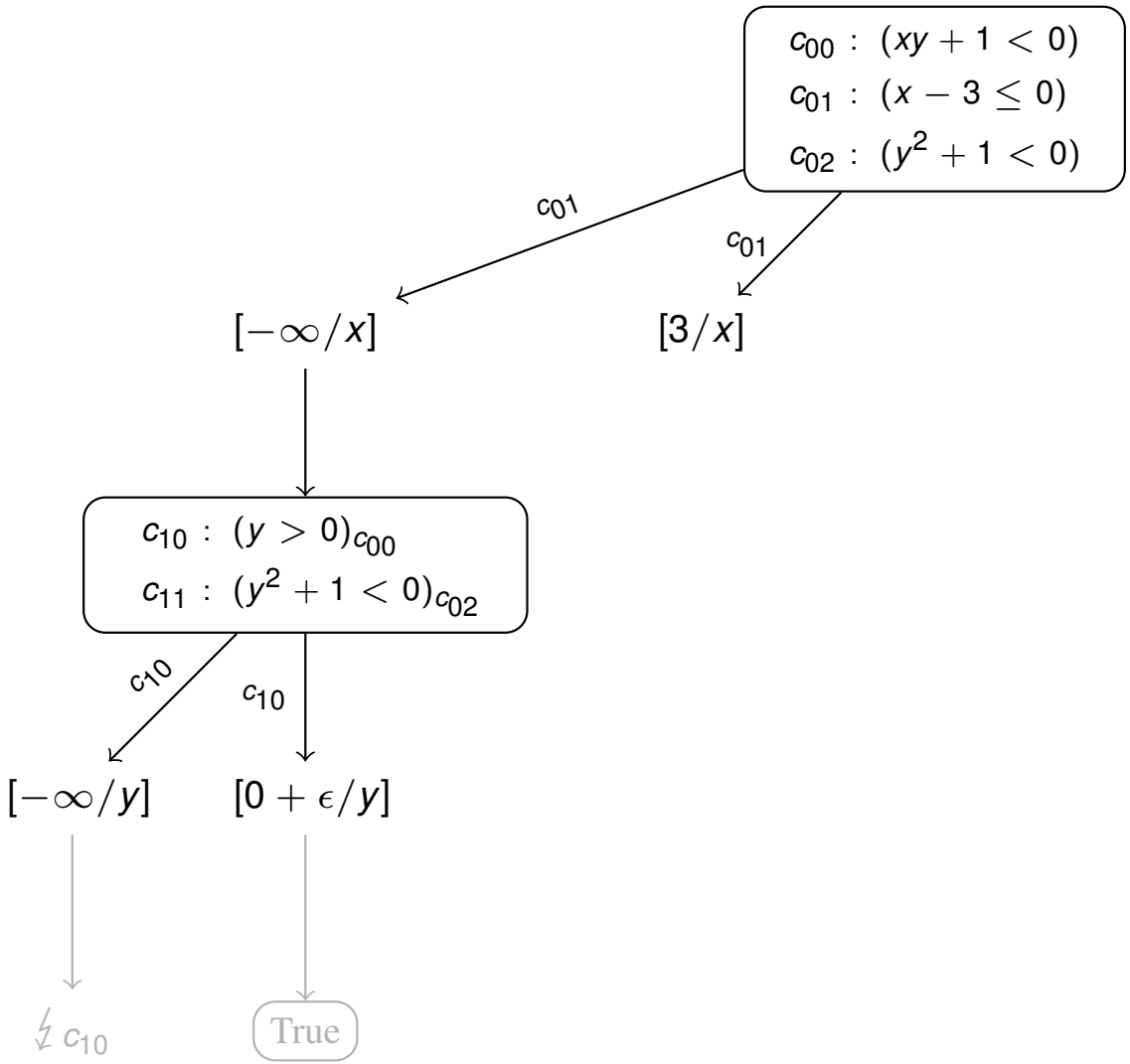

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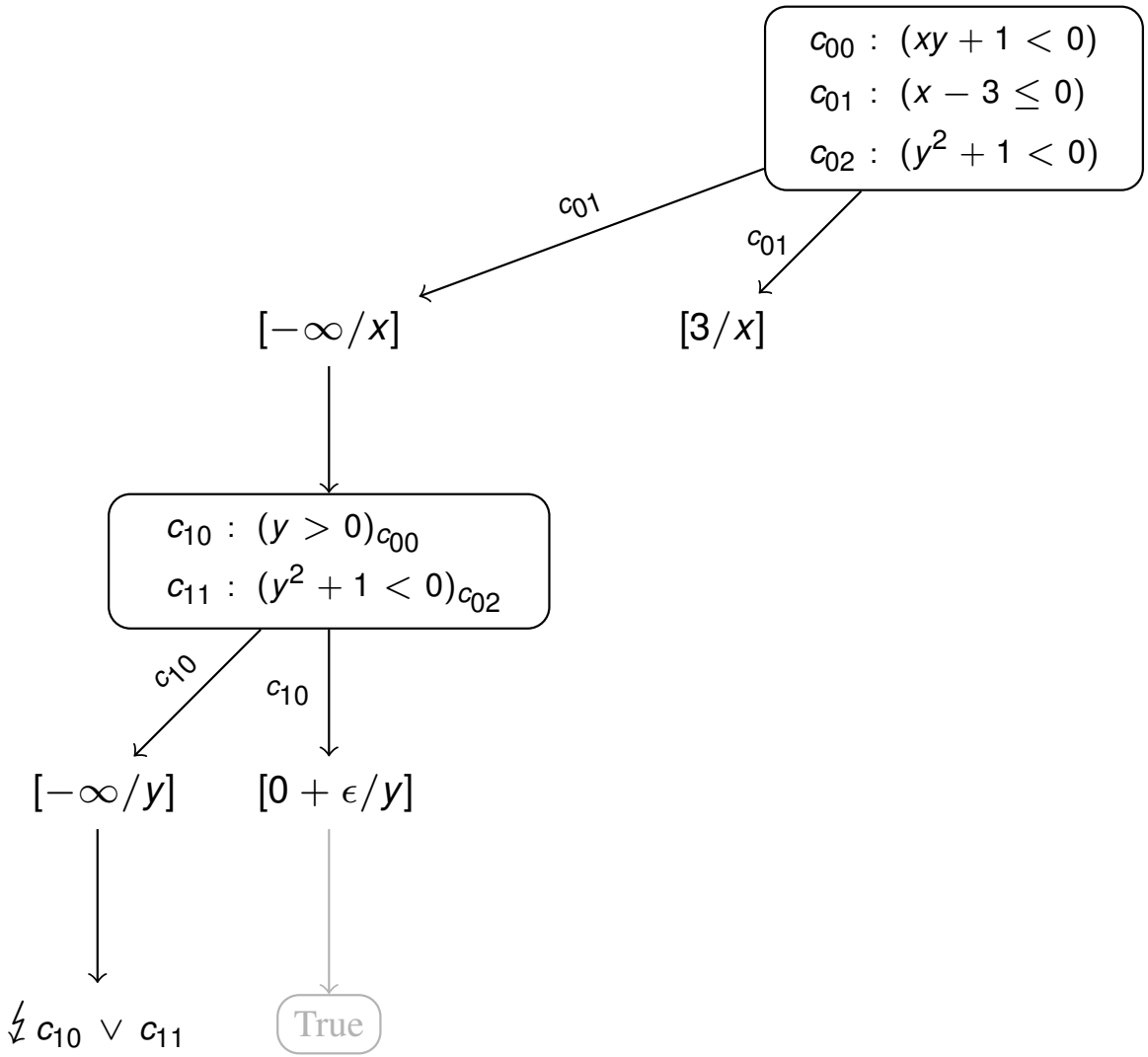

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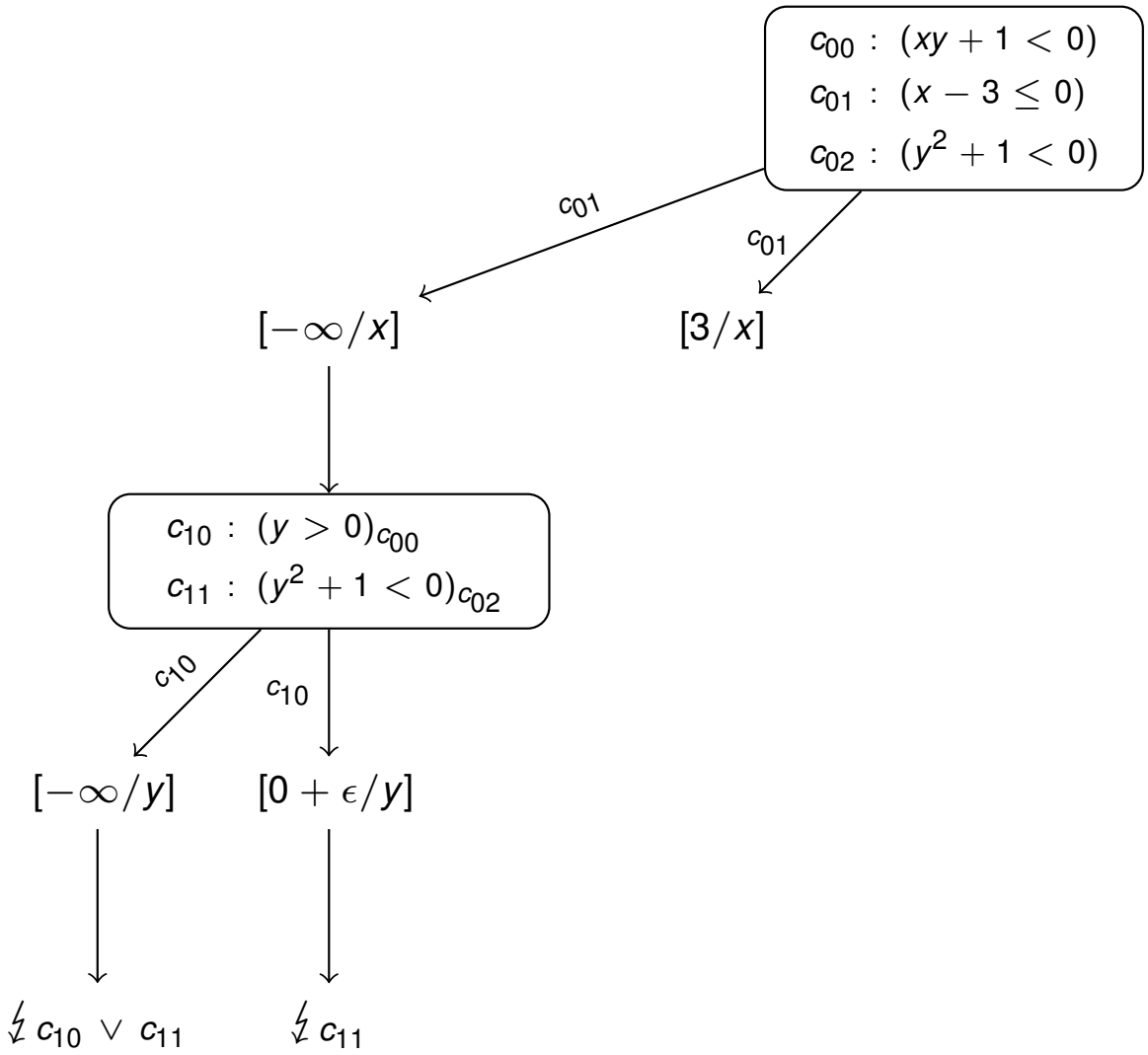

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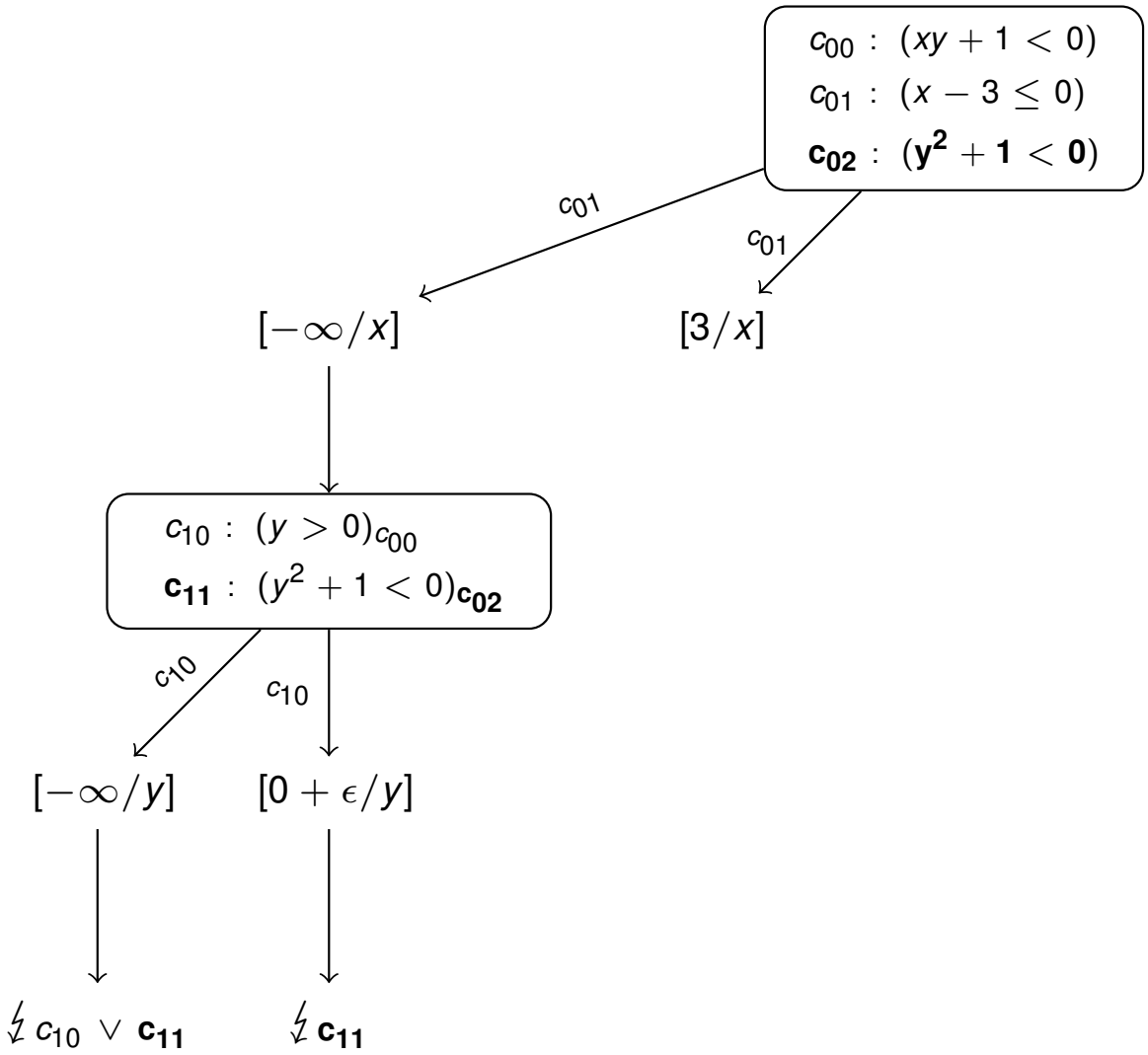

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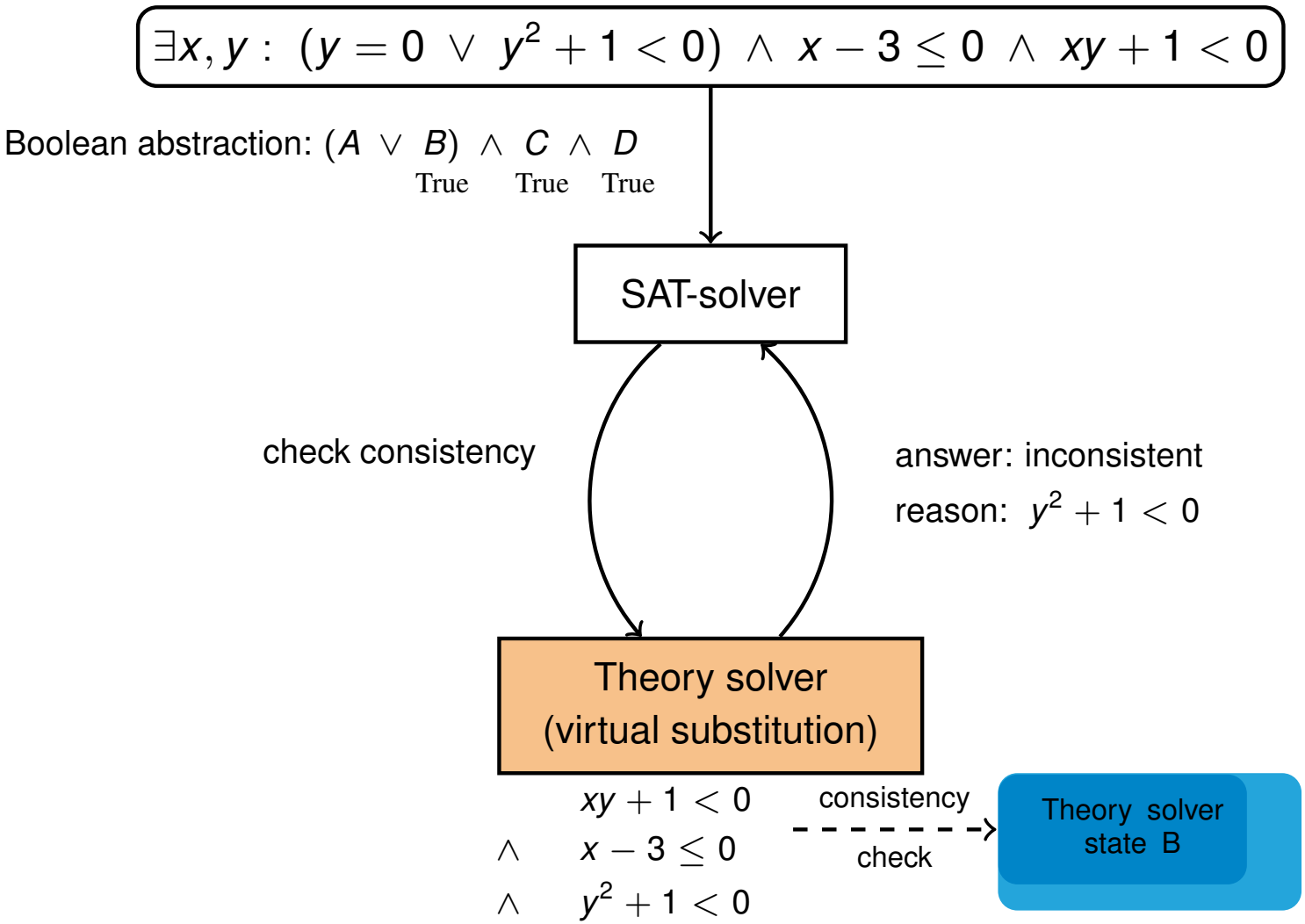
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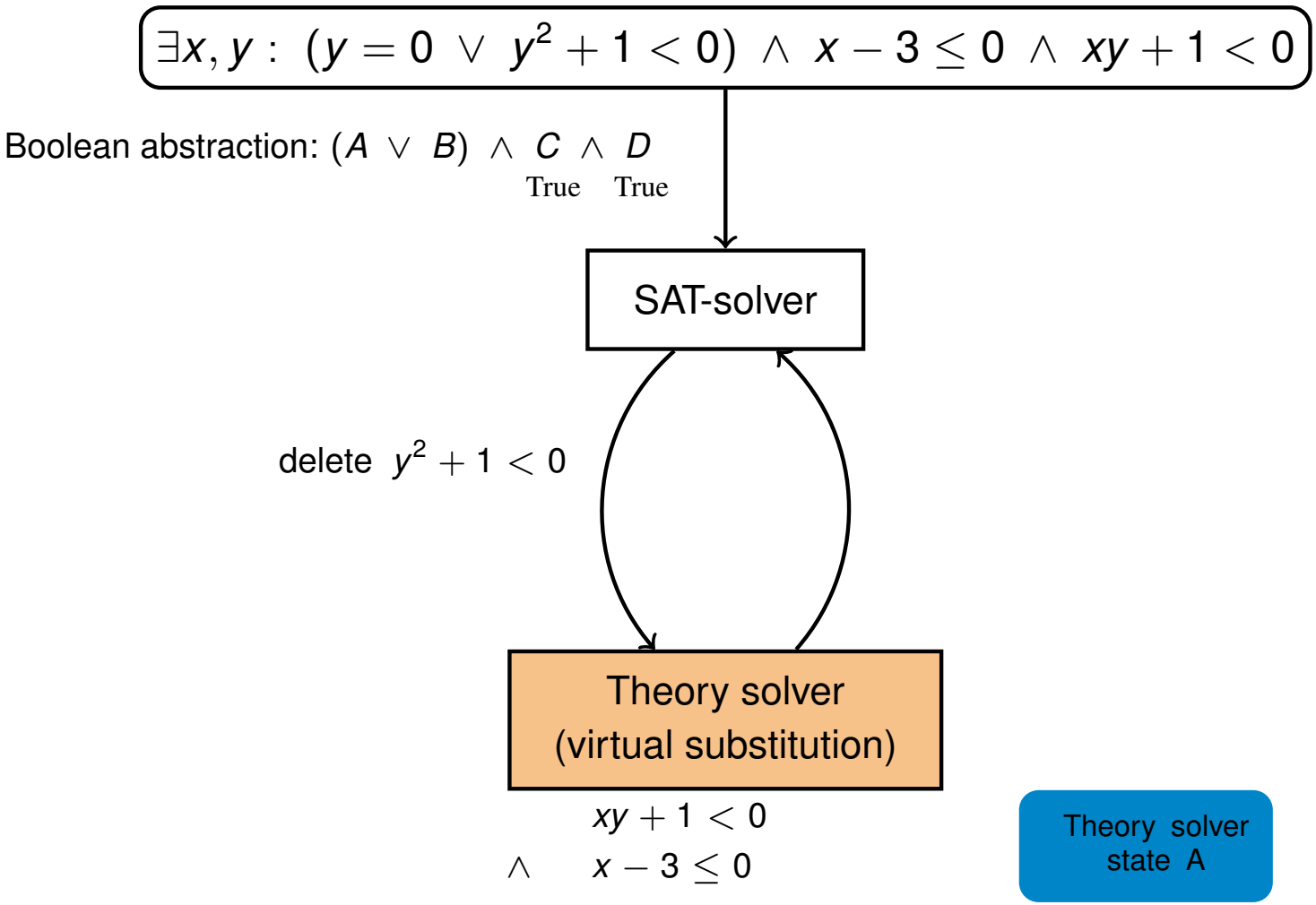
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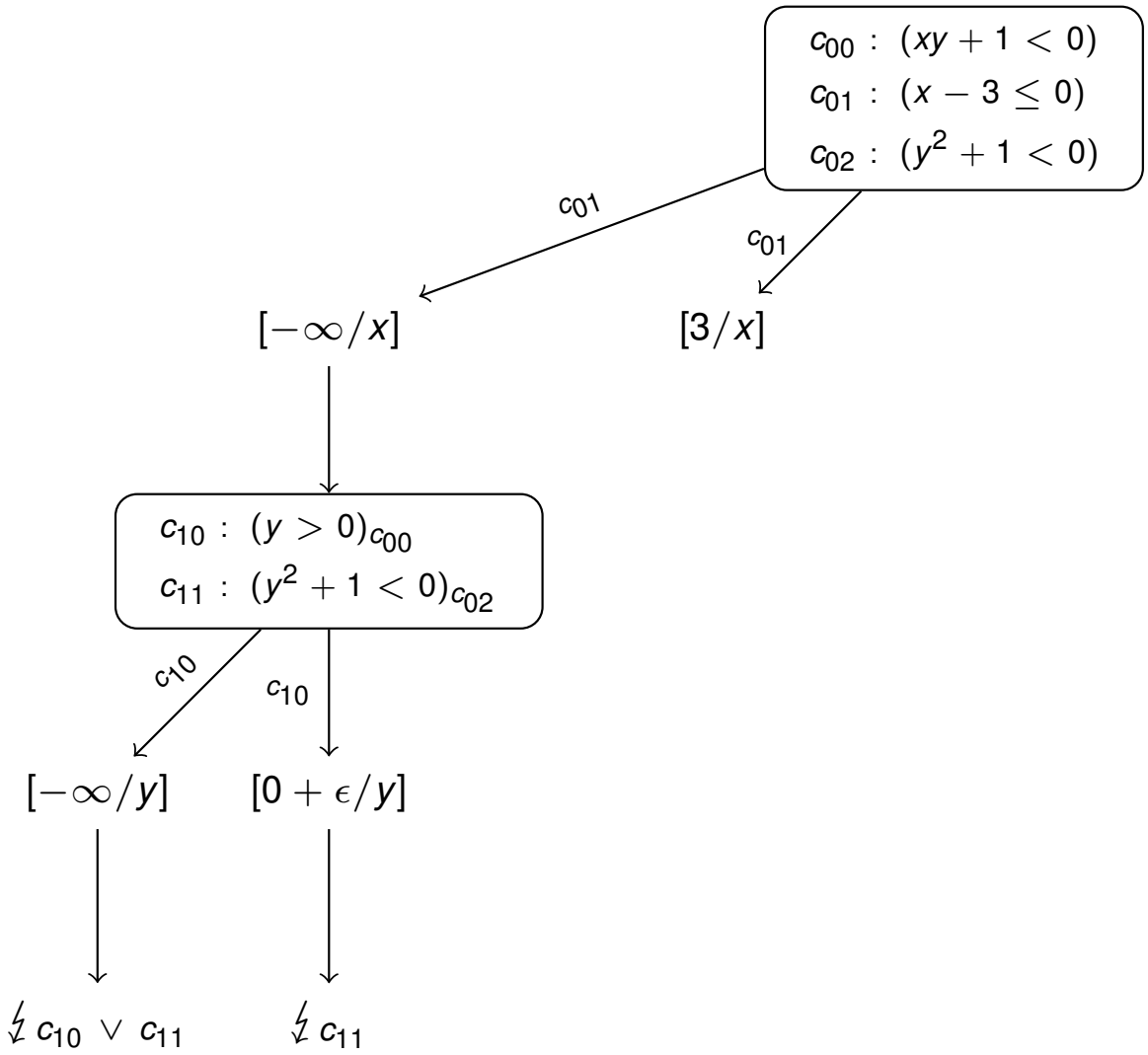

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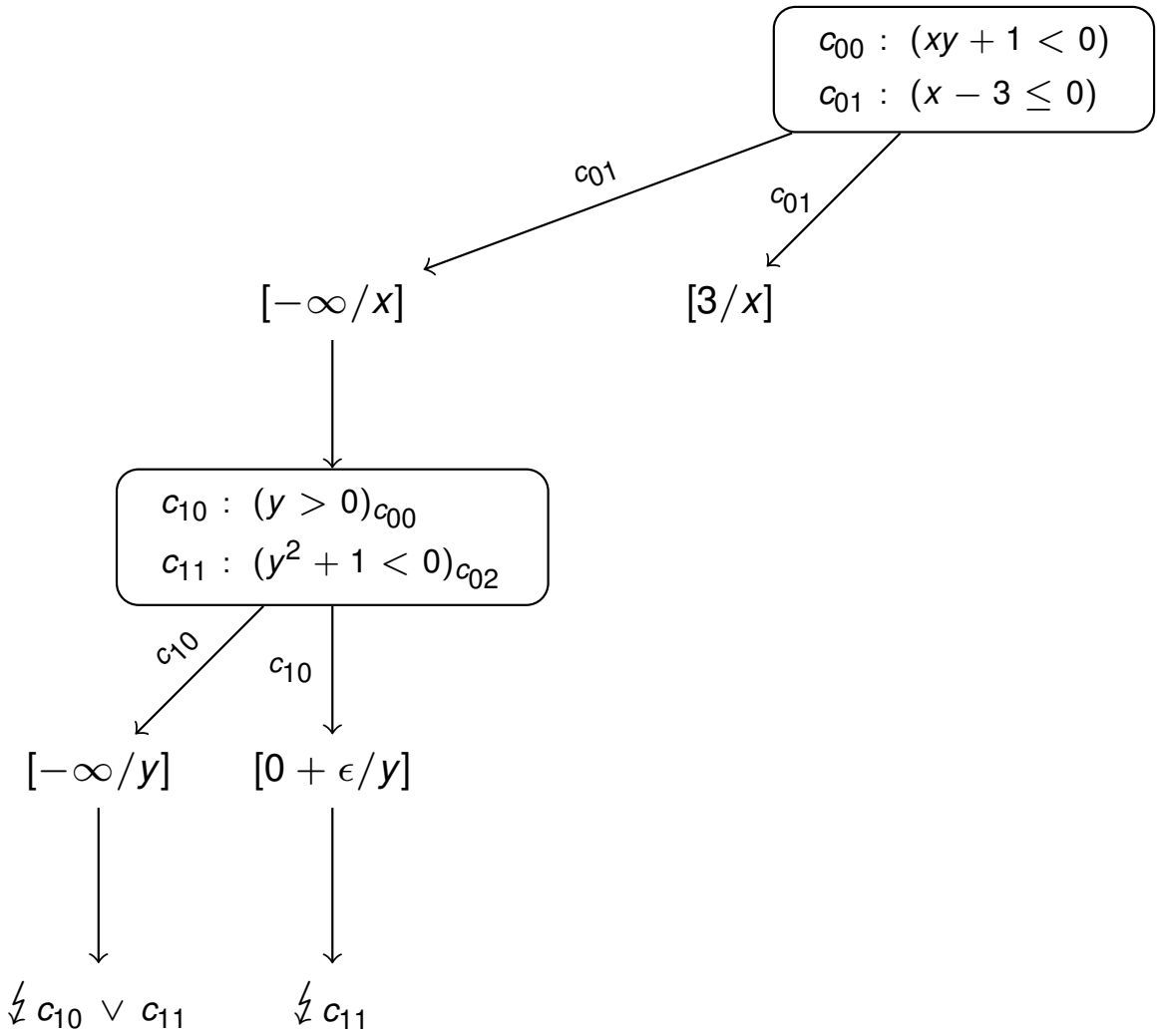

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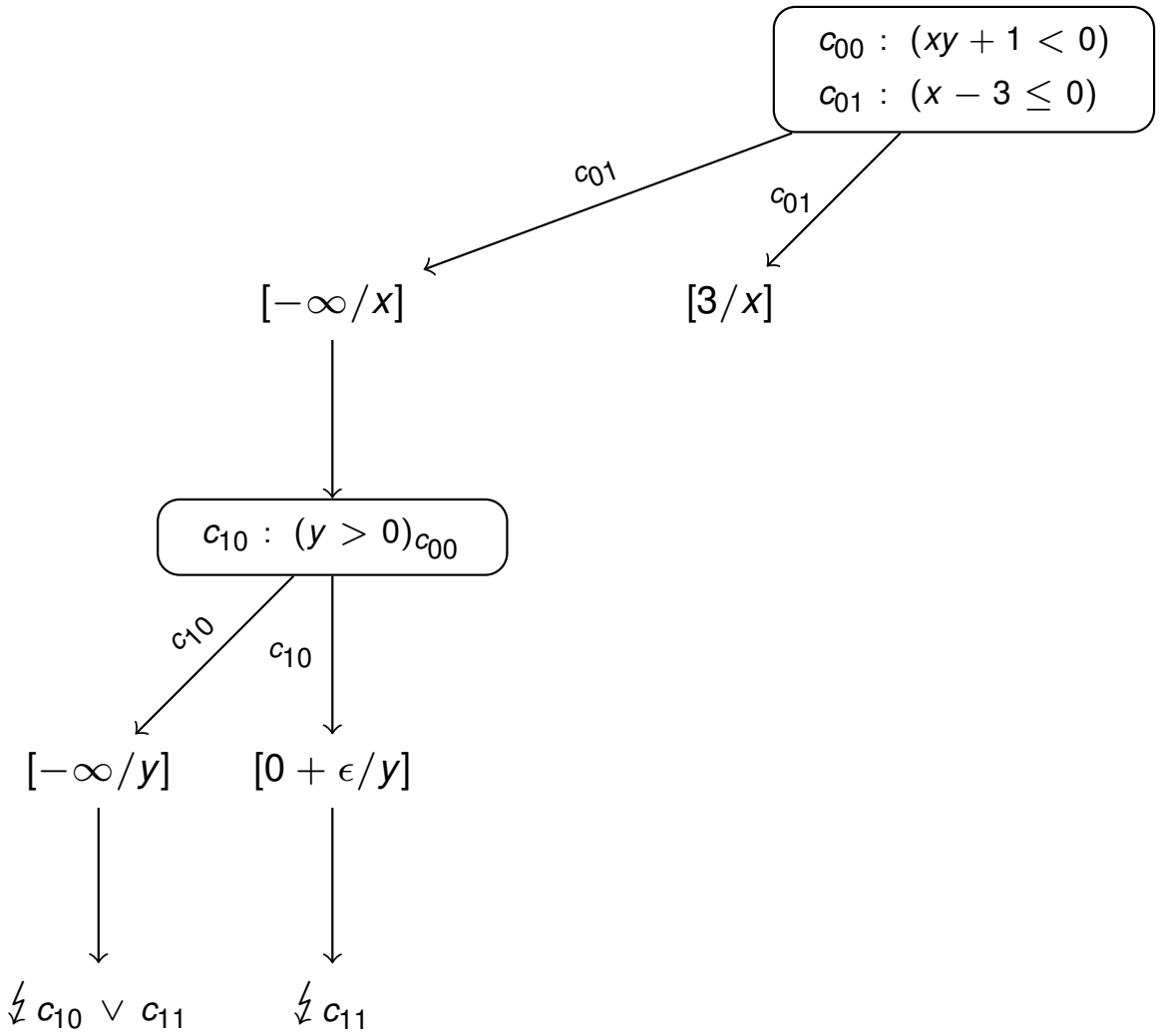

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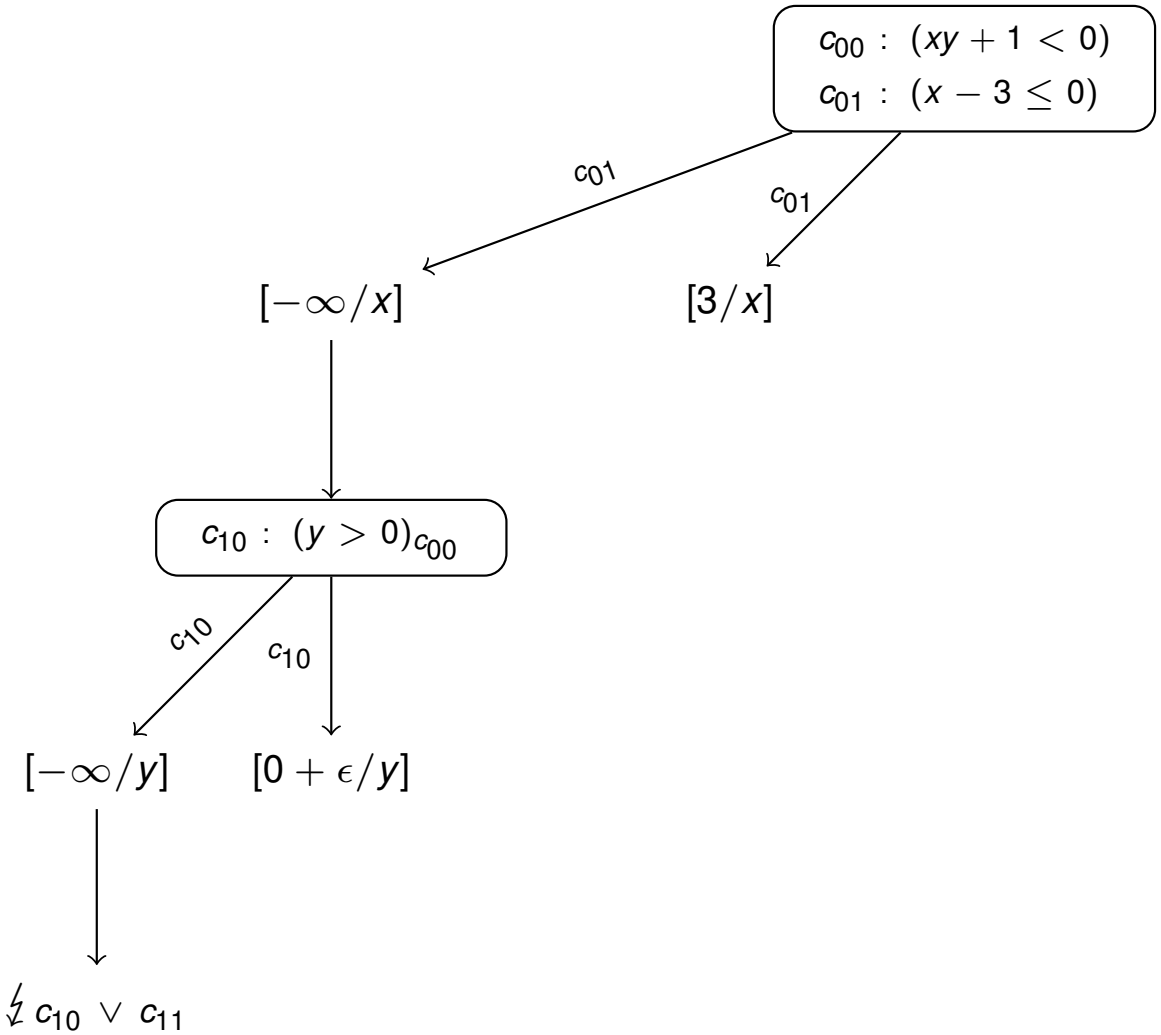

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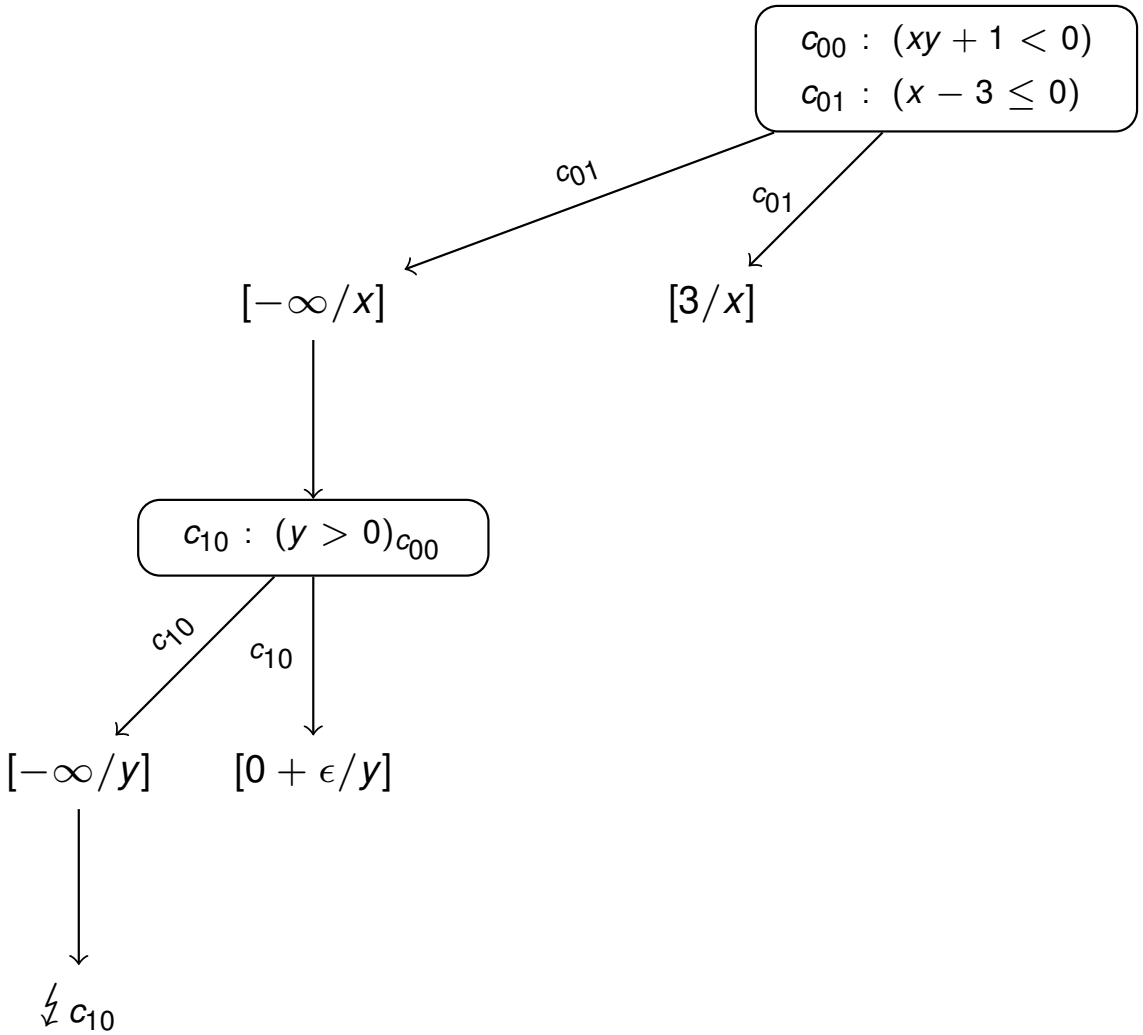

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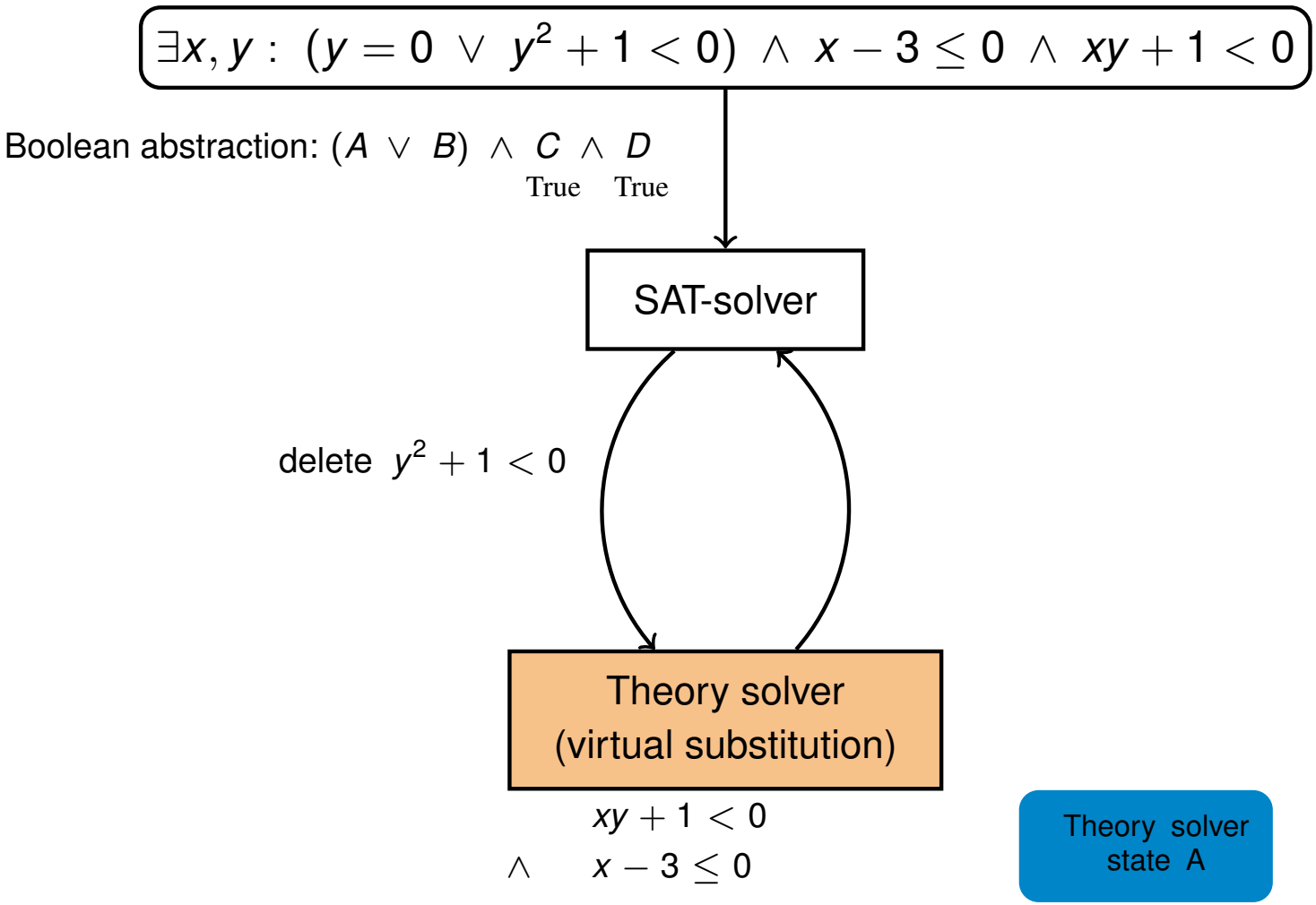
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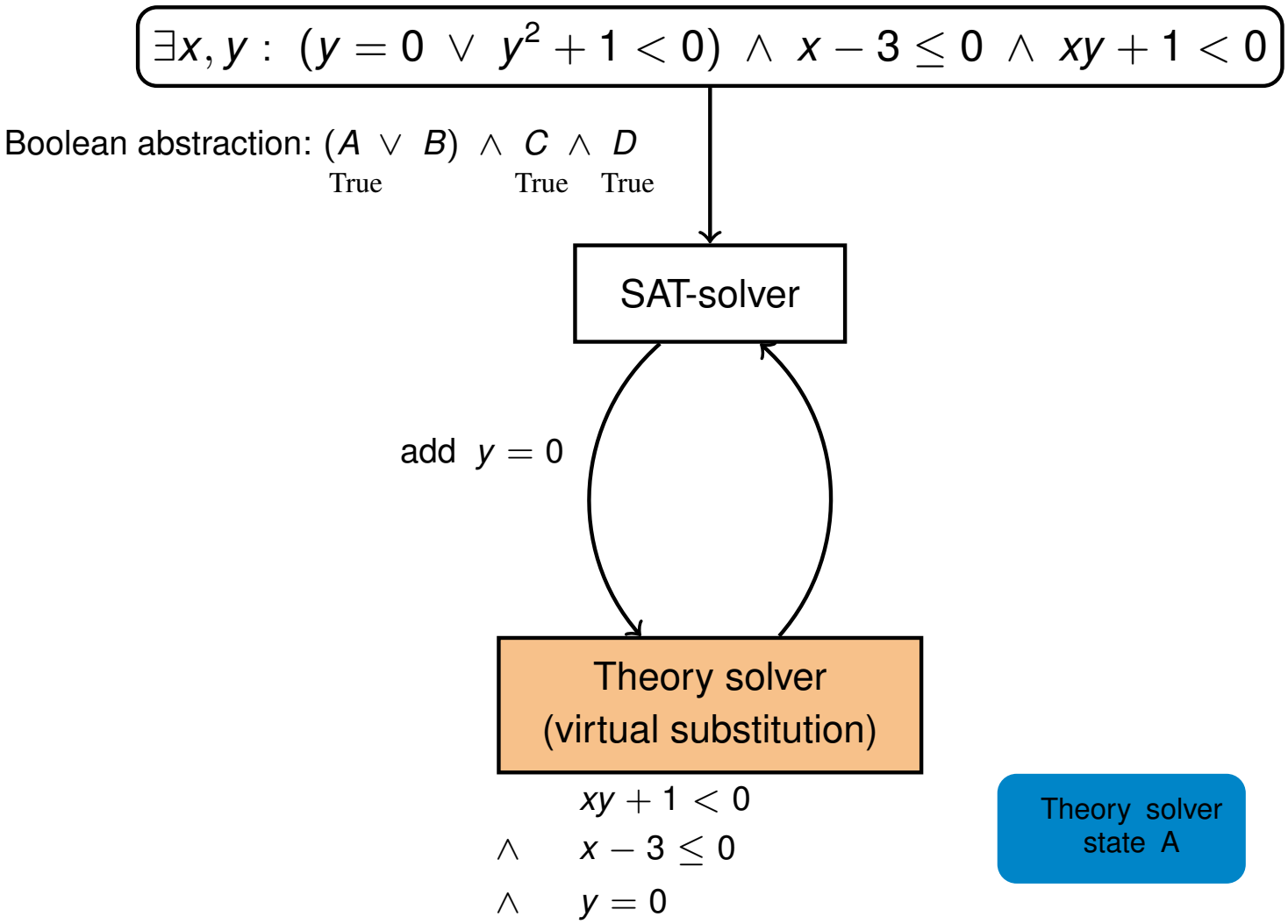
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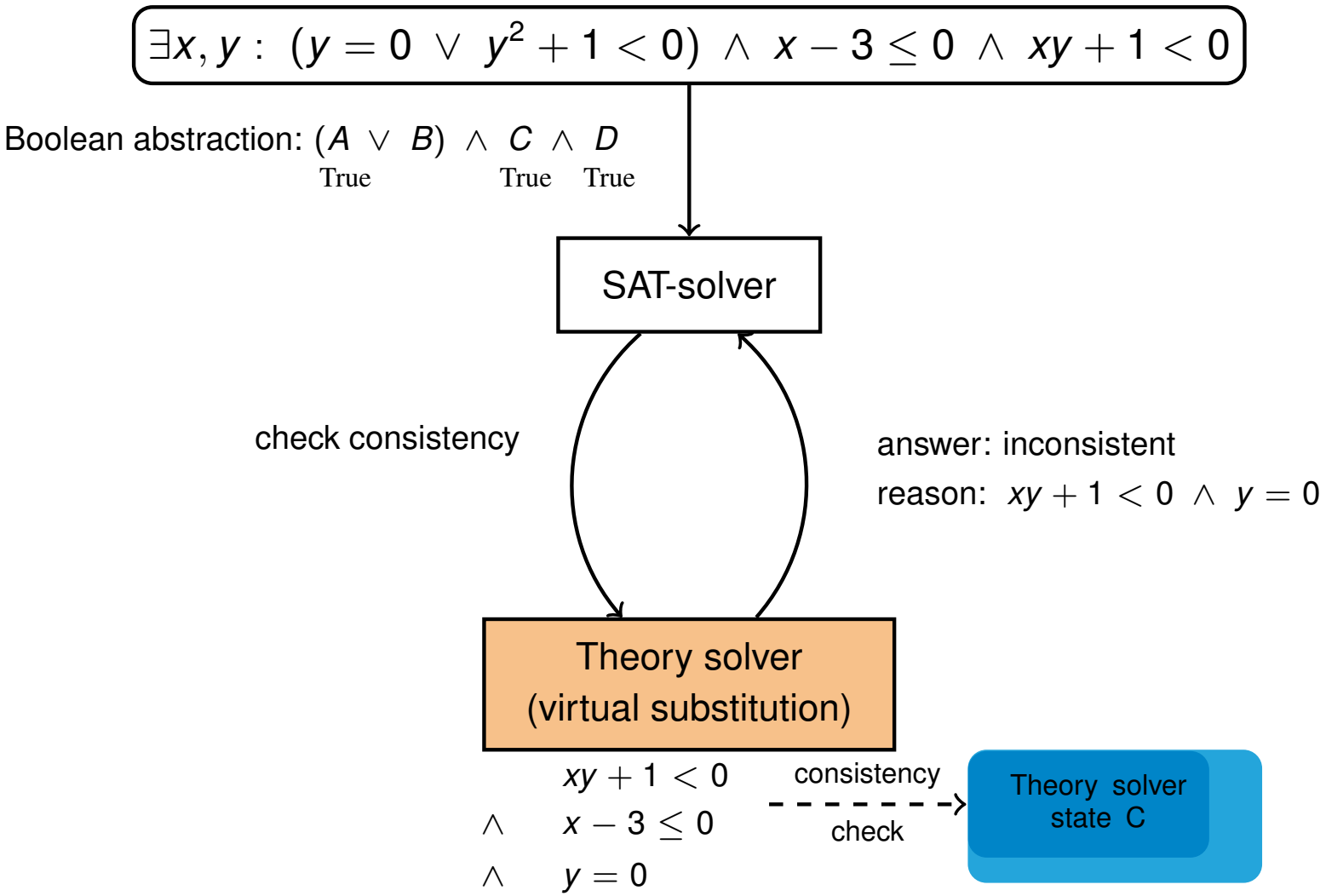
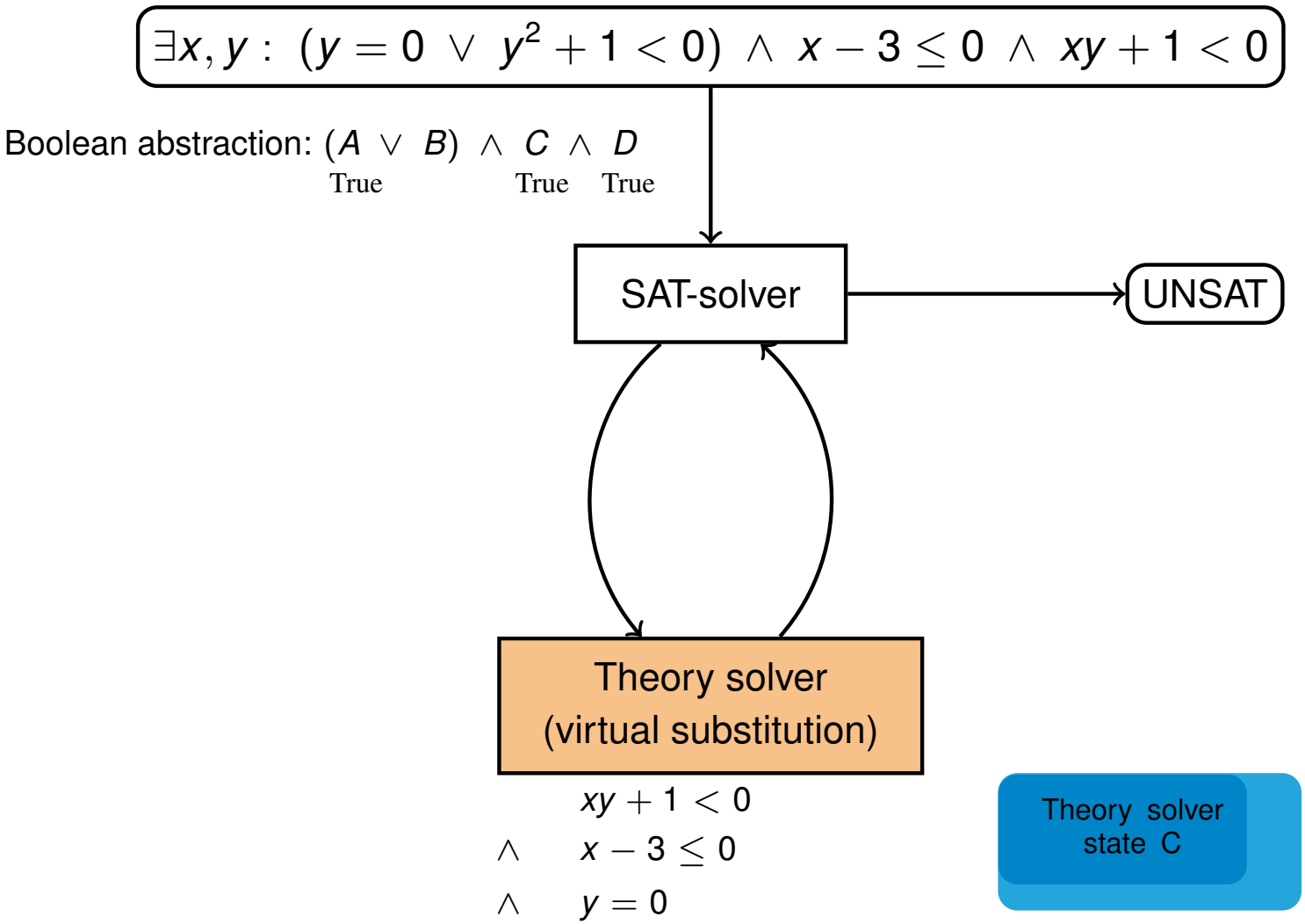
$xy + 1 < 0$
$\wedge \quad x - 3 \leq 0$
$\wedge \quad y = 0$
Theory solver state A

SMT-solver


$\exists x, y: \ (y = 0 \ \vee \ y^2 + 1 < 0) \ \wedge \ x - 3 \leq 0 \ \wedge \ xy + 1 < 0$
Boolean abstraction: $(A \ \vee \ B) \ \wedge \ C \ \wedge \ D$
True
True
True
SAT-solver
check consistency
answer: inconsistent
reason: $xy + 1 < 0 \ \wedge \ y = 0$
Theory solver
(virtual substitution)
$xy + 1 < 0$
$\wedge \quad x - 3 \leq 0$
$\wedge \quad y = 0$
consistency
check
Theory solver
state C

SMT-solver


$\exists x, y: \ (y = 0 \ \vee \ y^2 + 1 < 0) \ \wedge \ x - 3 \leq 0 \ \wedge \ xy + 1 < 0$
Boolean abstraction: $(A \ \vee \ B) \ \wedge \ C \ \wedge \ D$
True
True
True
SAT-solver
UNSAT
Theory solver
(virtual substitution)
$xy + 1 < 0$
$\wedge \quad x - 3 \leq 0$
$\wedge \quad y = 0$
Theory solver
state C

Example formula:

- 11 clauses each having 10 randomly generated constraints
- 20 variables

- Compared to the elaborated tool Redlog using virtual substitution

	Ø Runtime
Non-incremental with backjumping:	17.858s
Incremental:	33.576s
Non-incremental:	81.867s
Redlog:	1338.701s

- Achieved:
  - Theory solver using virtual substitution
    1. Incremental
    2. Supports backtracking
    3. Generates reasons (infeasible subsets)
  - Implementation of a theory solver
  - Embedding it into an SMT-solver

- Achieved:
  - Theory solver using virtual substitution
    1. Incremental
    2. Supports backtracking
    3. Generates reasons (infeasible subsets)
  - Implementation of a theory solver
  - Embedding it into an SMT-solver
- Future work:
  - Enable elimination of constraints with degree > 2 and ≤ 4
  - Interaction with the realizable sign conditions method
  - Optimize, optimize, optimize