Comparative Branching-Time Semantics for Markov Chains

Christel Baier[a], Joost-Pieter Katoen[bc], Holger Hermanns[dc] and Verena Wolf[e]

[a]University of Bonn, Römerstraße 164, D-53117 Bonn, Germany

[b]RWTH Aachen, Ahornstraße 55, D-52074 Aachen, Germany

[c]University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

[d]Saarland University, D-66123 Saarbrücken, Germany

[e]University of Mannheim, D-68131 Mannheim, Germany

This paper presents various semantics in the branching-time spectrum of discrete-time and continuous-time Markov chains (DTMCs and CTMCs). Strong and weak bisimulation equivalence and simulation pre-orders are covered and are logically characterised in terms of the temporal logics PCTL (Probabilistic Computation Tree Logic) and CSL (Continuous Stochastic Logic). Apart from presenting various existing branching-time relations in a uniform manner, this paper presents the following new results: (i) strong simulation for CTMCs, (ii) weak simulation for CTMCs and DTMCs, (iii) logical characterizations thereof (including weak bisimulation for DTMCs), (iv) a relation between weak bisimulation and weak simulation equivalence, and (v) various connections between equivalences and pre-orders in the continuous- and discrete-time setting. The results are summarized in a branching-time spectrum for DTMCs and CTMCs elucidating their semantics as well as their relationship.

Key words: comparative semantics, Markov chain, (weak) simulation, (weak) bisimulation, temporal logic

1. Introduction

To compare the stepwise behaviour of states in labeled transition systems, simulation ($\precsim$) and bisimulation relations ($\sim$) have been widely considered. Bisimulation relations are equivalences requiring two bisimilar states to exhibit identical stepwise behaviour [52–54]. On the contrary, simulation relations are preorders on the state space requiring that whenever $s \precsim s'$ ("s' simulates s") state s' can mimic all stepwise behaviour of s; the converse, i.e., $s' \precsim s$ is not guaranteed, so state s' may perform steps that cannot be matched by s. Thus, if s' simulates s then every successor of s has a corresponding, i.e., related successor of s', but the reverse does not necessarily hold. Simulation can be lifted to entire transition systems by comparing (according to $\precsim$) their initial states. Simulation relations are often used for verification purposes to show that one system correctly implements another, more abstract system [1,44,36,51,53]. One of the interesting aspects of simulation relations is that they allow a verification by "local" reasoning. The transitivity of $\precsim$ allows a stepwise verification in which the correctness is established via several intermediate systems. Simulation relations are therefore used as a basis for abstraction techniques where the rough idea is to replace the model to be verified by a smaller abstract model and to verify the latter instead of the original one. Typically, strong and weak bisimulation and simulation relations are distinguished. Whereas in *strong* (bi)simulations, each individual step needs to be mimicked, in *weak* (bi)simulations this is only required for observable steps but not for internal computations. Weak relations thus allow for stuttering.

A plethora of strong and weak (bi)simulations for labeled transition systems has been defined in the literature, and their relationship has extensively been studied by process algebraists, most notably by van Glabbeek [29,30]. These "comparative" semantics have been extended with logical characterizations that are of importance for verification purposes. Here, bisimulation relations possess the so-called *strong preservation* property, whereas simulation possesses *weak preservation*. Strong preservation means: if $s \sim s'$, then for all formulas Φ it follows $s \models \Phi$ iff $s' \models \Phi$. This property holds, for instance, for CTL (and CTL*) and strong bisimulation [18]. The use of simulation relies on the preservation of certain classes of formulas, not of all formulas (such as for $\sim$). For instance, if $s \precsim s'$ then for all safety (or even $\forall$CTL* [20]) formula Φ it follows that $s' \models \Phi$ implies $s \models \Phi$. Note that the converse, $s' \not\models \Phi$, cannot be used to deduce that Φ does not hold in the simulated state s; hence, the name *weak* preservation. Similar characterization results hold for branching (bi)simulation with divergence, a special type of weak (bi)simulation where typically the next operator is omitted, which is not compatible with stuttering. As simulation equivalence – defined as mutual simulation of states – is coarser than bisimulation equivalence it yields a "better abstraction", i.e., a smaller quotient.

For probabilistic systems, the situation is similar. Based on the seminal works of Jonsson and Larsen [45] and Larsen and Skou [50], notions of (bi)simulation (see, e.g., [3,9,15,17,31,37,39,46,56,58,62]) for models with and without nondeterminism have been defined during the last decade, and various logics to reason about such systems have been proposed (see e.g., [2,5,13,35]). This holds for both discrete probabilistic systems and variants thereof, as well as systems that describe continuous-time stochastic phenomena. In particular, in the discrete setting several slight variants of (bi)simulations have been defined, and their logical characterizations studied, e.g., [4,24,27,28,58]. Although the relationship between (bi)simulations is fragmentarily known, a clear, concise classification is lacking. To the best of our knowledge, simulation relations for the continuous-time setting have not been studied. Moreover, continuous-time and discrete-time semantics have largely been developed in isolation, and their connection has received scant attention, if at all.

This paper studies the comparative semantics of branching-time relations for probabilistic systems that do not exhibit any nondeterminism. In particular, time-abstract (or discrete-time) fully probabilistic systems (DTMCs) and continuous-time Markov chains (CTMCs) are considered. CTMCs are an important class of stochastic processes that are widely used in practice to determine system performance and dependability characteristics. Strong and weak (bi)simulation relations are covered together with their characterisation in terms of the temporal logics Probabilistic Computation Tree Logic (PCTL [35]) and Continuous Stochastic Logic (CSL [5,13]) for the discrete and continuous setting, respectively. PCTL is a discrete-probabilistic variant of CTL in which existential and universal path quantification have been replaced by a probabilistic path operator. PCTL allows one to specify properties like “the probability to reach a set of goal states via a restricted set of states is at least 0.74”, and is supported by efficient model-checking algorithms. CSL is a continuous probabilistic variant of CTL and includes means to express both transient and steady-state performance measures. For instance, it allows one to stipulate that the probability of reaching a certain set of goal-states within a specified real-valued time bound, provided that all paths to these states obey certain properties, is at least/at most some probability value. Model-checking algorithms for CSL have been presented in [8], and prototypical software implementations are available: for instance, one based on sparse matrices [38] and a symbolic tool based on multi-terminal binary decision diagrams [47].

Apart from presenting various existing branching-time relations and their connection in a uniform manner, this paper provides several new results:

- we propose a notion of weak simulation for CTMCs and show that this pre-order preserves (among others) probabilities on timed reachability properties. More precisely, the preorder weakly preserves a safe (live) fragment of CSL without next.
- as a side result, notions of strong simulation for CTMCs and weak simulation for DTMCs are established. These notions are shown to strongly preserve a safe fragment of CSL and weakly preserve a safe fragment of PCTL without next, respectively.
- weak bisimulation [9] for DTMCs is shown to coincide with equivalence for PCTL without next, and weak bisimulation [17] for CTMCs is shown to coincide with equivalence for CSL without next.
- weak (bi)simulation on CTMCs is shown to be invariant under uniformization [33, 41], and
- weak probabilistic bisimulation and weak simulation equivalence are shown to coincide, both for CTMCs and DTMCs.

Finally, several new relations are established between pre-orders and equivalences in the continuous-time and the discrete-time setting yielding a branching-time spectrum for CTMCs and DTMCs in the style of van Glabbeek.

Organization of the paper

The paper is further organized as follows. Section 2 provides the necessary background on Markov chains. Section 3 defines strong and weak (bi)simulations. Section 4 introduces PCTL and CSL and presents the logical characterizations. Section 5 summarizes the resulting branching-time spectrum and concludes the paper.

This paper is an extended version of the conference papers [12] and [11].

2. Markov chains

This section introduces the basic concepts of the Markov models considered within this paper; for a more elaborate treatment on such model see e.g., the textbooks [34,48,49].

2.1. Discrete-time probabilistic systems

Let AP be a fixed, finite set of atomic propositions. A fully probabilistic system is a Kripke structure where each transition is labeled with a discrete probability. Formally,

Definition 1. A *fully probabilistic system* (FPS) is a tuple $\mathcal{D} = (S, \mathbf{P}, L)$ where:

- S is a countable set of states
- $\mathbf{P} : S \times S \to [0, 1]$ is a probability matrix satisfying $\sum_{s' \in S} \mathbf{P}(s, s') \in [0, 1]$ for all $s \in S$
- $L : S \to 2^{AP}$ is a labeling function which assigns to each state $s \in S$ the set $L(s)$ of atomic propositions that are (assumed to be) valid in s. ■

If $\sum_{s' \in S} \mathbf{P}(s, s') = 1$, state s is called stochastic; if this sum equals zero, i.e., if $\mathbf{P}(s, s') = 0$ for all s', state s is called absorbing; otherwise, s is called sub-stochastic. A discrete-time Markov chain (DTMC) is an FPS such that for any state the sum of the probabilities of its emanating transitions is either zero or one.

Definition 2. A (labeled) DTMC is an FPS where any state is either stochastic or absorbing, i.e., $\sum_{s' \in S} \mathbf{P}(s, s') \in \{ 0, 1 \}$ for all $s \in S$. ■

For $C \subseteq S$, $\mathbf{P}(s, C) = \sum_{s' \in C} \mathbf{P}(s, s')$ denotes the probability for s to move to a state in C. For technical reasons, $\mathbf{P}(s, \bot) = 1 - \mathbf{P}(s, S)$. Intuitively, $\mathbf{P}(s, \bot)$ denotes the probability to stay forever in s without performing any transition; although $\bot$ is not a "real" state (i.e., $\bot \notin S$), it may be regarded as a deadlock. In the context of simulation relations later on, $\bot$ is treated as an auxiliary state that is simulated by any other state. Let $S_\bot = S \cup \{ \bot \}$. $\mathsf{Post}(s) = \{ s' \mid \mathbf{P}(s, s') > 0 \}$ denotes the set of direct successor states of s, and

$$\mathsf{Post}_\bot(s) = \{ s' \in S_\bot \mid \mathbf{P}(s, s') > 0 \} = \mathsf{Post}(s) \cup \{ \bot \mid \mathbf{P}(s, \bot) > 0 \}.$$

Note that the following statements hold:

$$\begin{array}{llllllll} s \text{ is stochastic} & \text{iff} & \bot \notin \mathsf{Post}_\bot(s) & \text{iff} & \mathbf{P}(s,\bot) = 0 & \text{iff} & \mathbf{P}(s, S) = 1 \text{ and} \\ s \text{ is absorbing} & \text{iff} & \mathsf{Post}_\bot(s) = \{ \bot \} & \text{iff} & \mathbf{P}(s,\bot) = 1 & \text{iff} & \mathbf{P}(s, S) = 0. \end{array}$$

2.2. Continuous-time Markov chains

We consider FPSs and therefore also DTMCs as *time-abstract* models. The name DTMC has historical reasons. A (discrete-)timed interpretation is appropriate in settings where all state changes occur at equidistant time points. In contrast, CTMCs are considered as *time-aware*, as they have an explicit reference to time, in the form of transition rates which determine the stochastic evolution of the system in time.

Definition 3. A (labeled) CTMC is a tuple $\mathcal{C} = (S, \mathbf{R}, L)$ with S and L as before, and *rate matrix* $\mathbf{R} : S \times S \to \mathbb{R}_{\geqslant 0}$ such that $\sum_{s' \in S} \mathbf{R}(s, s')$ converges. The value $E(s) = \sum_{s' \in S} \mathbf{R}(s, s')$ is called the exit rate of state s. ■

As in the discrete case, $\mathsf{Post}(s) = \{ s' \mid \mathbf{R}(s, s') > 0 \}$ denotes the set of direct successor states of s, and for $C \subseteq S$, $\mathbf{R}(s, C) = \sum_{s' \in C} \mathbf{R}(s, s')$ denotes the rate of moving from state s to a state in C via a single transition. Note $E(s) = \mathbf{R}(s, S)$. State s in a CTMC is called absorbing if $E(s) = 0$.

Intuitively, the rates specify the average delays of the transitions. More precisely, the meaning of $\mathbf{R}(s, s') = \lambda > 0$ is that with probability $1 - e^{-\lambda \cdot t}$ the transition $s \to s'$ is enabled within the next t time units provided that the current state is s. If $\mathbf{R}(s, s') > 0$ for more than one state s', a *race* between the outgoing transitions from s exists. The probability of s' winning this race before time t is determined as follows:

$$\Pr\{s \to s' \text{ wins the race before time } t \mid \text{the system is in state } s \text{ at time } 0\}$$

$$= \int_0^t \underbrace{\mathbf{R}(s, s') \cdot e^{-\mathbf{R}(s,s') \cdot x}}_{\substack{\text{density function of} \\ \text{the distribution for } s \to s'}} \cdot \underbrace{\prod_{s'' \in \mathsf{Post}(s) \setminus \{s'\}} e^{-\mathbf{R}(s,s'') \cdot x}}_{\substack{\text{probability that the earliest time} \\ \text{at which } s \to s'' \text{ can fire exceeds } x}} \, dx$$

$$= \int_0^t \mathbf{R}(s, s') \cdot e^{-E(s) \cdot x} \, dx \;=\; \frac{\mathbf{R}(s, s')}{E(s)} \cdot (1 - e^{-E(s) \cdot t})$$

With $t \to \infty$ we get from the above calculations that $\mathbf{R}(s, s')/E(s)$ denotes the probability that the delay of going from s to s' "finishes before" the delays of any other outgoing transition from s. Summing over all states $s' \in \mathsf{Post}(s)$ (i.e., independent outcomes) we obtain:

$$\sum_{s' \in S} \frac{\mathbf{R}(s, s')}{E(s)} \cdot (1 - e^{-E(s) \cdot t}) \;=\; \frac{E(s)}{E(s)} \cdot (1 - e^{-E(s) \cdot t}) \;=\; 1 - e^{-E(s) \cdot t}$$

is the probability to take an outgoing transition from state s within the next t time units [1].

The time-abstract probabilistic behaviour of CTMC $\mathcal{C}$ is described by its so-called embedded DTMC:

Definition 4. The *embedded* DTMC of CTMC $\mathcal{C} = (S, \mathbf{R}, L)$ is given by $emb(\mathcal{C}) = (S, \mathbf{P}, L)$, where $\mathbf{P}(s, s') = \mathbf{R}(s, s')/E(s)$ if $E(s) > 0$ and $\mathbf{P}(s, s') = 0$ otherwise. ■

Note that, by definition, the embedded DTMC $emb(\mathcal{C})$ of any CTMC $\mathcal{C}$ does not contain sub-stochastic states.

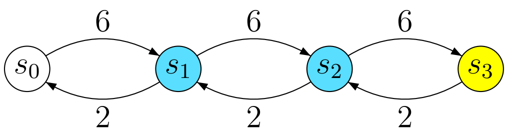
A CTMC is called *uniformized* if all its states have the same exit rate, i.e., $E(s) = E(s')$ for all states s, s'. The embedded DTMC of a uniformized CTMC does not contain absorbing states (except if E=0). Each CTMC can be transformed into a uniformized CTMCs by adding self-loops [57]:

[1] This explains the notion "exit rate" E. However, as we allow for self-loops (i.e., states s with $\mathbf{R}(s, s) > 0$) as e.g., in [57,8], "leaving" state s includes that the self-loop $s \to s$ (if any) maybe taken.

Definition 5. Let $\mathcal{C} = (S, \mathbf{R}, L)$ be a CTMC and let (uniformisation rate) E be a real such that $E \geqslant max_{s \in S}\ E(s)$. Then, $unif(\mathcal{C}) = (S, \overline{\mathbf{R}}, L)$ is a uniformized CTMC with $\overline{\mathbf{R}}(s, s') = \mathbf{R}(s, s')$ for $s \neq s'$, and $\overline{\mathbf{R}}(s, s) = \mathbf{R}(s, s) + E - E(s)$. ■

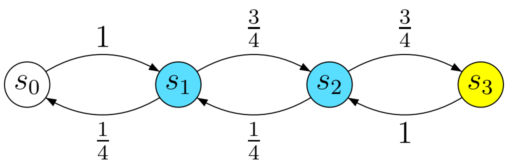
The minimal appropriate value of E is determined by the state in $\mathcal{C}$ with the shortest mean residence time. (Strictly speaking, we should write $unif_E(\mathcal{C})$ as the uniformization depends on E.) In $unif(\mathcal{C})$ all rates of self-loops are "normalized" with respect to E. As a result, state transitions occur with an average "pace" of E, uniform for all states of the chain. We will later see that $\mathcal{C}$ and $unif(\mathcal{C})$ are related by weak bisimulation. Note that in the literature [33,41], uniformisation is often defined as a transformation of CTMC $\mathcal{C}$ into the DTMC $emb(unif(\mathcal{C}))$. For technical convenience, we here define uniformisation as a CTMC-to-CTMC transformation (as e.g., in [57]) by basically adding self-loops to "slower" states.

Example 6.



The figure just above illustrates a CTMC that models a queuing system with a buffer capacity of three items and where the arrival and departure rate of items is 6 and 2, respectively. State s_i represents the configuration in which the queue contains i jobs ($0 \leqslant i < 4$). The shadings indicate the labeling of states, e.g., we assume that $AP = \{\ empty, full\}$ and that $L(s_0) = \{\ empty\}$, $L(s_1) = L(s_2) = \varnothing$, and $L(s_3) = \{\ full\}$.

The embedded DTMC of this queuing system is:

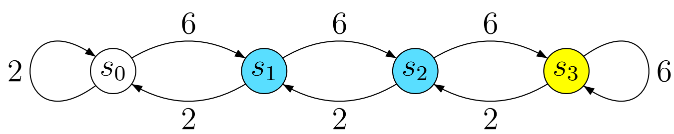


For instance, for state s_1, we have $E(s_1) = 6{+}2 = 8$ and

$$\mathbf{P}(s_1, s_2) = \mathbf{R}(s_1, s_2)/E(s_1) = 6/8 = 3/4,$$
$$\mathbf{P}(s_1, s_0) = \mathbf{R}(s_1, s_0)/E(s_1) = 2/8 = 1/4.$$

For state s_3, we have $E(s_3) = 2$ and $\mathbf{P}(s_3, s_2) = \mathbf{R}(s_3, s_2)/E(s_3) = 2/2 = 1$.

The uniformized CTMC of the queuing system for $E{=}8$ is:



As $E(s_0) < E$ and $E(s_3) < E$, states s_0 and s_3 in the original CTMC are left with a lower frequency than $\frac{1}{E}$, and are therefore equipped with a self-loop. According to the same principle, states s_1 and s_2 would be also equipped with a self-loop if $E > 8$.

3. Bisimulation and simulation

This section defines simulation pre-orders and bisimulation equivalences on FPSs and CTMCs, presents several basic results of these relations, and characterizes their relation. Strong relations are presented prior to their weak variants. We will use the subscript "d" to identify relations defined in the discrete setting (FPSs or DTMCs), and "c" for the continuous setting (CTMCs).

3.1. Strong bisimulation

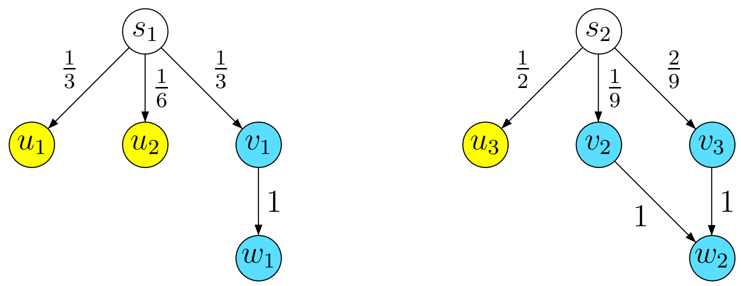
One of the most elementary equivalence relations on discrete-time probabilistic systems is probabilistic bisimulation [50]. This variant of strong bisimulation for labeled transition systems considers two states to be equivalent if the cumulative probability to move to any of the equivalence classes that this relation induces is the same. We consider a slight variant of the original notion in which we require in addition that equivalent states are equally labeled. This is exploited later to establish logical characterizations.

Definition 7. [**48,50,46,31**] Let $\mathcal{D} = (S, \mathbf{P}, L)$ be a FPS and R an equivalence relation on S. R is a *strong bisimulation* on $\mathcal{D}$ if for $s_1 \, R \, s_2$:

$$L(s_1) = L(s_2) \quad \text{and} \quad \mathbf{P}(s_1, C) = \mathbf{P}(s_2, C) \text{ for all } C \text{ in } S/R.$$

s_1 and s_2 in $\mathcal{D}$ are strongly bisimilar, denoted $s_1 \sim_d s_2$, if there exists a strong bisimulation R on $\mathcal{D}$ with $s_1 \, R \, s_2$. ■

Note that in any FPS we have: $s_1 \sim_d s_2$ implies $\mathbf{P}(s_1, \perp) = \mathbf{P}(s_2, \perp)$.



Example 8. *States s_1 and s_2 of the FPS depicted just above (where equally shaded states are labeled with the same atomic propositions) are strongly bisimilar. To prove this, it suffices to show that the equivalence R which identifies the two s-states, the three u-states, the three v-states and the two w-states, satisfies the conditions in Def. 7. The labeling condition is obviously fulfilled as R identifies equally-labeled states. Furthermore, note that all u- and w-states are absorbing, and hence, $\mathbf{P}(u_i, C) = \mathbf{P}(w_j, C) = 0$ (for $0 < i \leqslant 3$ and $0 < j \leqslant 2$) for each R-equivalence class C. For the s-states, we have: $\mathbf{P}(s_1, \{ u_1, u_2, u_3 \}) = \frac{1}{2} = \mathbf{P}(s_2, \{ u_1, u_2, u_3 \})$ and $\mathbf{P}(s_1, \{ v_1, v_2, v_3 \}) = \frac{1}{3} = \mathbf{P}(s_2, \{ v_1, v_2, v_3 \})$. Moreover, $\mathbf{P}(v_i, \{ w_1, w_2 \}) = 1$ (for $0 < i \leqslant 2$). Thus, R is a strong bisimulation containing (s_1, s_2), and hence $s_1 \sim_d s_2$.* ■

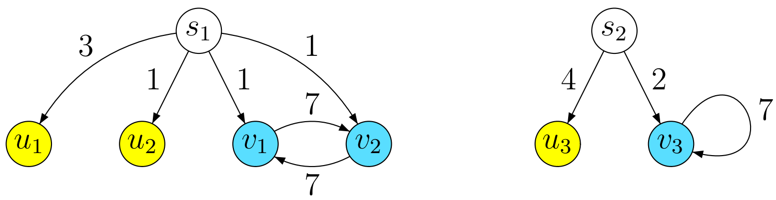
Strong bisimulation for CTMCs, also known as ordinary lumpability, is a mild variant of the notion for the discrete-time probabilistic setting where it is required that the cumulative rate (instead of the discrete probability) for two equivalent states to move to any of the induced equivalence classes is equal.

Definition 9. **[19,39]** Let $\mathcal{C} = (S, \mathbf{R}, L)$ be a CTMC and R an equivalence relation on S. R is a *strong bisimulation* on $\mathcal{C}$ if for $s_1 \, R \, s_2$:

$$L(s_1) = L(s_2) \quad \text{and} \quad \mathbf{R}(s_1, C) = \mathbf{R}(s_2, C) \text{ for all } C \text{ in } S/R.$$

s_1 and s_2 in $\mathcal{C}$ are strongly bisimilar, denoted $s_1 \sim_c s_2$, if there exists a strong bisimulation R on $\mathcal{C}$ with $s_1 \, R \, s_2$. ■

Example 10. *Consider the CTMC depicted below. The relation R identifying the two s-states, the three u-states and the two v-states, is a strong bisimulation on CTMCs, as $\mathbf{R}(s_1, \{ u_1, u_2 \}) = 4 = \mathbf{R}(s_2, \{ u_3 \})$, $\mathbf{R}(s_1, \{ v_1, v_2 \}) = 2 = \mathbf{R}(s_2, \{ v_3 \})$, the u-states are absorbing, and $\mathbf{R}(v_i, \{ v_1, v_2, v_3 \}) = 7$ for $0 < i \leqslant 3$. As $(s_1, s_2) \in R$, it follows $s_1 \sim_c s_2$.* ■



As $\mathbf{R}(s, C) = \mathbf{P}(s, C) \cdot E(s)$, the condition on the cumulative rates can be reformulated as

$$\mathbf{P}(s_1, C) = \mathbf{P}(s_2, C) \text{ for all } C \in S/R \quad \text{and} \quad E(s_1) = E(s_2).$$

Hence, $\sim_c$ agrees with $\sim_d$ in the embedded DTMC provided that the exit rates are treated as additional atomic propositions. From these observations it directly follows:

Proposition 11. For CTMC $\mathcal{C} = (S, \mathbf{R}, L)$:

1. *$s_1 \sim_c s_2$ implies $s_1 \sim_d s_2$ in* $\mathrm{emb}(\mathcal{C})$*, for any state $s_1, s_2 \in S$.*
2. *if $\mathcal{C}$ is uniformized then $\sim_c$ coincides with $\sim_d$ in* $\mathrm{emb}(\mathcal{C})$*.*

By the standard construction for bisimulation on labeled transition systems, it can be shown that $\sim_d$ and $\sim_c$ are the coarsest strong bisimulations.

3.2. Strong simulation

3.2.1. Weight functions

Definition 12. A *distribution* on set S is a function $\mu : S \to [0, 1]$ with $\sum_{s \in S} \mu(s) \leqslant 1$. ■

We put $\mu(\perp) = 1 - \sum_{s \in S} \mu(s)$. Let *Distr*$(S)$ denote the set of all distributions on S. Distribution μ on S is called *stochastic* if $\mu(\perp) = 0$. For labeled transition systems, state s' simulates state s if for each successor state t of s there is a one-step successor state t' of s' that simulates t. Simulation of two states is thus defined in terms of simulation of their successor states. (It is therefore sometimes called forward simulation.) In the probabilistic setting, the target of a transition is in fact a probability distribution, and thus, the simulation relation $\precsim$ needs to be lifted from states to distributions. In fact, strong bisimulation on FPSs was defined as an equivalence on S such that all R-equivalent states s_1 and s_2 are equally labeled and

$$\mathbf{P}(s_1, \cdot) \;\equiv_R\; \mathbf{P}(s_2, \cdot)$$

where $\equiv_R$ denotes the lifting of R on $Distr(S)$ defined as:

$$\mu \equiv_R \mu' \quad \text{iff} \quad \mu(C) = \mu'(C) \text{ for all } C \in S/R.$$

(It is easy to see that $\equiv_R$ is an equivalence.) The rough idea behind the definition of simulation relations is to replace the equivalence $\equiv_R$ by a non-symmetric relation $\sqsubseteq_R$ which is obtained using the concept of weight functions.

Definition 13. [43,45] Let S be a set, $R \subseteq S \times S$, and $\mu, \mu' \in Distr(S)$. A *weight function* for μ and μ' with respect to R is a function $\Delta : S_\perp \times S_\perp \to [0, 1]$ such that:

1. $\Delta(s, s') > 0$ implies $s\, R\, s'$ or $s = \perp$
2. $\mu(s) = \sum_{s' \in S_\perp} \Delta(s, s')$ for any $s \in S_\perp$
3. $\mu'(s') = \sum_{s \in S_\perp} \Delta(s, s')$ for any $s' \in S_\perp$

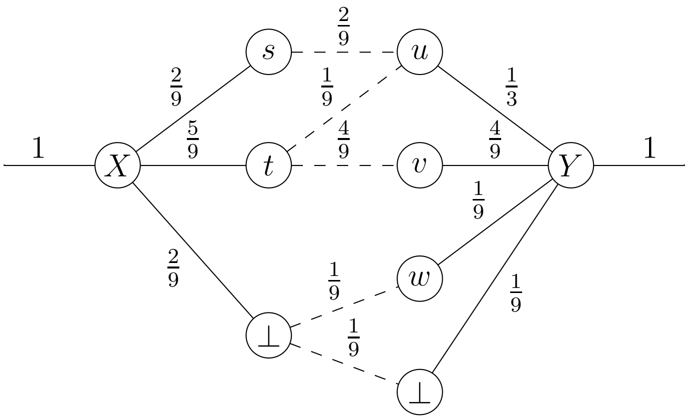
We write $\mu \sqsubseteq_R \mu'$ (or simply $\sqsubseteq$, if R is clear from the context) iff there exists a weight function for μ and μ' with respect to R. $\sqsubseteq_R$ is the lift of R to distributions. ■

Intuitively, Δ distributes a probability distribution over a set X to a distribution over a set Y such that the total probability assigned by Δ to $y \in Y$ equals the original probability $\mu'(y)$ on Y. In a similar way, the total probability mass of $x \in X$ that is assigned by Δ must coincide with the probability $\mu(x)$ on X. Δ is a probability distribution on $X \times Y$ such that the probability to select (x, y) with $x\, R\, y$ is one. In addition, the probability to select an element in R whose first component is x equals $\mu(x)$, and the probability to select an element in R whose second component is y equals $\mu'(y)$. For any state y, Δ may assign a positive probability to $\perp$. Hence, the deadlock symbol $\perp$ is treated as a "bottom state" that is simulated by any other state (independent of the labeling).

Example 14. *Let* $S = \{\, s, t, u, v, w \,\}$ *with* $\mu(s) = \frac{2}{9}$, $\mu(t) = \frac{5}{9}$ *and* $\mu'(u) = \frac{1}{3}$, $\mu'(v) = \frac{4}{9}$, $\mu'(w) = \frac{1}{9}$ *and* $\mu(\cdot) = \mu'(\cdot) = 0$ *for the remaining cases. Note that* μ *and* μ' *are both sub-stochastic. Let*

$$R \;=\; \{\, (s, u), (t, u), (t, v) \,\}.$$

We have $\mu \sqsubseteq_R \mu'$, *as, e.g., weight function* Δ *(see picture below where, for convenience,* $\perp$ *is depicted as a state) defined by* $\Delta(s, u) = \frac{2}{9}, \Delta(t, u) = \frac{1}{9}, \Delta(t, v) = \frac{4}{9}, \Delta(\perp, w) = \frac{1}{9}$, *and* $\Delta(\perp, \perp) = \frac{1}{9}$ *satisfies the constraints of Def. 13.*



■

Note that $\Delta(s,\bot) = 0$ for all states $s \in S$ whereas $\Delta(\bot,\bot)$ may be positive. Moreover:

$$\mu(S) \;=\; \sum_{s\in S}\sum_{s'\in S}\Delta(s,s') \;=\; \sum_{s'\in S}\sum_{s\in S}\Delta(s,s') \;\leqslant\; \sum_{s'\in S}\sum_{s\in S_\bot}\Delta(s,s') \;=\; \mu'(S).$$

From this, it follows that whenever μ is stochastic then so is μ', i.e., if $\mu(S) = 1$ then $\mu'(S) = 1$ and $\Delta(\bot, s') = 0$ for all $s' \in S_\bot$. Hence, in this case Δ can be viewed as a stochastic distribution on $S \times S$. In particular, for stochastic distributions the concept of weight functions is symmetric, provided R is symmetric. The same holds for distributions μ, μ' where $\mu(S) = \mu'(S) \leqslant 1$. This yields the second part of the following proposition. The proof of the third part can be provided with the help of flow functions in networks [42,23,7]. The proof of the first part is straightforward.

Proposition 15. **[45,6,23]** *Let S be a set and $R \subseteq S \times S$.*

1. *If R is reflexive (transitive) then so is $\sqsubseteq_R$.*
2. *If R is symmetric and $\mu, \mu' \in Distr(S)$ with $\mu(S) = \mu'(S)$ then*

$$\mu \sqsubseteq_R \mu' \;\; \textit{iff} \;\; \mu' \sqsubseteq_R \mu.$$

3. *If R is an equivalence on S and $\mu, \mu' \in Distr(S)$ with $\mu(S) = \mu'(S)$ then*

$$\mu \equiv_R \mu' \;\; \textit{iff} \;\; \mu \sqsubseteq_R \mu'.$$

In particular, $\sqsubseteq_R$ as a binary relation on the set of stochastic distributions is an equivalence and agrees with $\equiv_R$.

3.2.2. The discrete-time setting

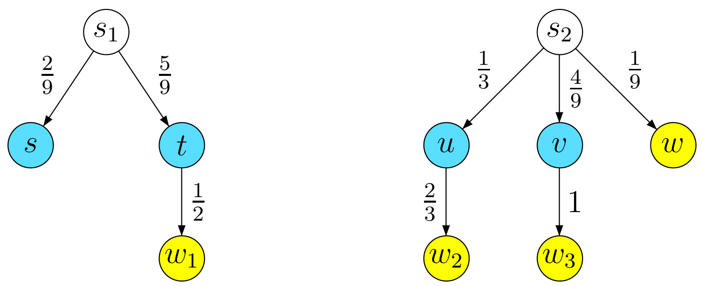
Given the notion of weight functions, we now will present how such functions can be used to define simulation relations. In the discrete-time setting, simulating states need to be equally labeled, and a weight function must exist that relates their one-step probabilities. Formally,

Definition 16. [45] Let $\mathcal{D} = (S, \mathbf{P}, L)$ be a FPS and $R \subseteq S \times S$. R is a *strong simulation* on $\mathcal{D}$ if for all $s_1 \, R \, s_2$:

$$L(s_1) = L(s_2) \quad \text{and} \quad \mathbf{P}(s_1, \cdot) \sqsubseteq_R \mathbf{P}(s_2, \cdot).$$

s_2 strongly simulates s_1 in $\mathcal{D}$, denoted $s_1 \precsim_d s_2$, iff there exists a strong simulation R on $\mathcal{D}$ such that $s_1 \, R \, s_2$. ■

Example 17.



In the FPS depicted above, $s_1 \precsim_d s_2$ as the relation

$$R = \{ (s_1, s_2), (s, u), (t, u), (t, v), (w_1, w_2), (w_1, w_3) \}$$

is a strong simulation. A weight function for the one-step successors of s_1 and s_2 w.r.t. R was presented in Example 14. ■

From Prop. 15.1 it follows that $\precsim_d$ is a preorder.

Remark. It can be shown that $\precsim_d$ is the coarsest strong simulation on $\mathcal{D}$. In particular, if $s_1 \precsim_d s_2$ then $\mathbf{P}(s_1, \cdot) \sqsubseteq \mathbf{P}(s_2, \cdot)$ where $\sqsubseteq$ denotes the lifting of $\precsim_d$ to distributions. The same will hold for the other simulation relations we define on FPSs and CTMCs. This will not be explicitly stated anymore. ■

By Prop. 15.3 we directly obtain:

Proposition 18. [45]

1. *$s_1 \sim_d s_2$ implies $s_1 \precsim_d s_2$.*
2. *For any DTMC without absorbing states, $\precsim_d$ is symmetric and coincides with $\sim_d$.*

Note that $\precsim_d$ is non-symmetric for DTMCs that may have absorbing states, as, e.g, any absorbing state s_1 is strongly simulated by any state s_2 with $L(s_1) = L(s_2)$ while the converse does *not* hold. However, strong simulation equivalence (i.e., the kernel of $\precsim_d$) agrees with $\sim_d$. This result can be shown using an alternative characterisation of strong simulations by means of the upward- or downward closure of subsets of states. These closures are defined as follows.

Definition 19. Let S be a set, $C \subseteq S$, and $R \subseteq S \times S$ be a pre-order. Then:

$$\begin{aligned} C \uparrow_R &= \{ s' \in S \mid s \, R \, s' \text{ for some } s \in C \}, \\ C \downarrow_R &= \{ s' \in S \mid s' \, R \, s \text{ for some } s \in C \}. \end{aligned}$$

C is *R-downward-closed* iff $C = C{\downarrow_R}$, and C is R-upward-closed iff $C = C{\uparrow_R}$. ■

$C{\uparrow_R}$ denotes the R-upward closure of C, whereas $C{\downarrow_R}$ stands for the R-downward closure of C. For $C = \{ s \}$, we simply write $s{\uparrow_R}$ and $s{\downarrow_R}$. If R is understood from the context, we simply write $C{\downarrow}$ and $C{\uparrow}$. Note that if R is an equivalence relation, then $s{\uparrow} = s{\downarrow} = [s]_R$, i.e., the equivalence class of s under R.

Proposition 20. **[14,6,23]** *For any FPS, $\precsim_d$ is the coarsest binary relation R on the state space S such that for all $s_1 R s_2$:*

$$L(s_1) = L(s_2) \quad \textit{and} \quad \mathbf{P}(s_1, C{\uparrow_R}) \leqslant \mathbf{P}(s_2, C{\uparrow_R}) \quad \textit{for all } C \subseteq S.$$

For $C \subseteq S$, C is downward-closed iff $S \setminus C$ is upward-closed. Moreover,

$$\mathbf{P}(s, C) \;=\; \mathbf{P}(s, S) - \mathbf{P}(s, S \setminus C) \;=\; 1 - \mathbf{P}(s, \perp) - \mathbf{P}(s, S \setminus C).$$

Hence, the second conjunct in Prop. 20 may be replaced by $\mathbf{P}(s_1, C{\downarrow_R} \cup \{ \perp \}) \geqslant \mathbf{P}(s_2, C{\downarrow_R} \cup \{ \perp \})$ for all $C \subseteq S$.

Proposition 21. **[6,23]** $\precsim_d \cap \precsim_d^{-1}$ *coincides with* $\sim_d$.

Proof: By Prop. 18.1, $\sim_d$ contains $\precsim_d \cap \precsim_d^{-1}$. We now show that $\precsim_d \cap \precsim_d^{-1}$ contains $\sim_d$. Let s_1 and s_2 be two strong simulation equivalent states of FPS $\mathcal{D} = (S, \mathbf{P}, L)$. Let B be the strong simulation equivalence class of s_1 (and s_2) and let $C_1 = B{\uparrow_{\precsim_d}}$ and $C_2 = C_1 \backslash B$. Then, C_1 and C_2 are upward-closed wrt. $\precsim_d$; hence, by Prop. 20, $\mathbf{P}(s_1, C_1) = \mathbf{P}(s_2, C_1)$ and $\mathbf{P}(s_1, C_2) = \mathbf{P}(s_2, C_2)$. Moreover, $\mathbf{P}(s_i, C_1) = \mathbf{P}(s_i, C_2) + \mathbf{P}(s_i, B)$ (for i=1, 2). Hence, $\mathbf{P}(s_1, B) = \mathbf{P}(s_2, B)$. So, $\precsim_d \cap \precsim_d^{-1}$ is a strong bisimulation and $s_1 \sim_d s_2$. ■

3.2.3. The continuous-time setting.

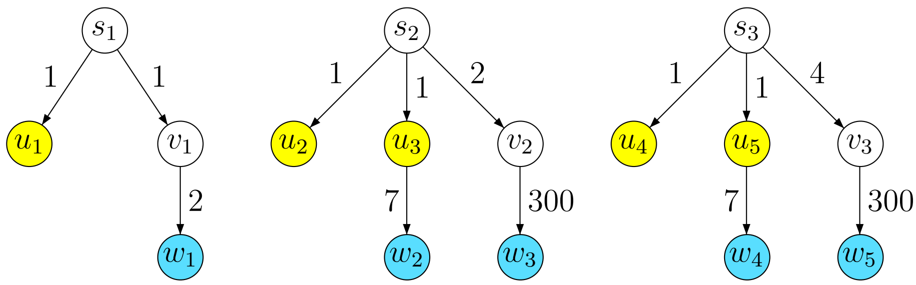
The intention of a simulation preorder on CTMCs is to ensure that state s_2 simulates s_1 if and only if (i) s_2 is "faster than" s_1 and (ii) the time-abstract behavior of s_2 simulates that of s_1. Note that compared to the discrete-time setting, the only extra requirement is the "faster than" constraint, the other constraints are identical. It therefore directly follows that this notion is a pre-order. Its formal definition is:

Definition 22. Let $\mathcal{C} = (S, \mathbf{R}, L)$ be a CTMC and $R \subseteq S \times S$. R is a *strong simulation* on $\mathcal{C}$ if for all $s_1 \, R \, s_2$:

$$L(s_1) = L(s_2), \quad \mathbf{P}(s_1, \cdot) \sqsubseteq_R \mathbf{P}(s_2, \cdot) \quad \text{and} \quad E(s_1) \leqslant E(s_2).$$

s_2 strongly simulates s_1 in $\mathcal{C}$, denoted $s_1 \precsim_c s_2$, iff there exists a strong simulation R on $\mathcal{C}$ such that $s_1 \, R \, s_2$. ■

Example 23.

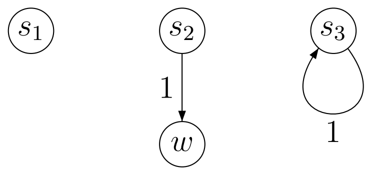


The above picture illustrates a CTMC where $s_1 \precsim_c s_2$ and $s_2 \not\precsim_c s_3$. This can be seen by checking the conditions of being a strong simulation. First, observe that these states are equally labeled (as indicated by their shading). Consider s_1 and s_2. The rate condition, i.e., the third condition of Def. 22, is obviously fulfilled as $E(s_1) = 2 \leqslant 4 = E(s_2)$. The weight function condition, i.e., the second condition, is fulfilled as

$$R \;=\; \{(s_1, s_2),\ (u_1, u_2),\ (u_1, u_3),\ (v_1, v_2),\ (w_1, w_2),\ (w_1, w_3)\ \}$$

can be shown to be a strong simulation. For (s_1, s_2), an appropriate weight function is: $\Delta(u_1, u_2) = \Delta(u_1, u_3) = \frac{1}{4}$, $\Delta(v_1, v_2) = \frac{1}{2}$, and $\Delta(\cdot) = 0$ otherwise. Accordingly, $s_1 \precsim_c s_2$. Consider s_2 and s_3. The rate condition for these states is fulfilled as $E(s_2) = 4 \leqslant 6 = E(s_3)$, but the distribution to move to the u- and v-states is different, e.g., $\mathbf{P}(s_2, \{ u_2, u_3 \}) = \frac{1}{2} \neq \frac{1}{3} = \mathbf{P}(s_3, \{ u_4, u_5 \})$. So, $s_2 \not\precsim_c s_3$. ■

Example 24.



In the above depicted CTMC we have $s_1 \precsim_c s_2$ and $s_2 \precsim_c s_3$, but $s_2 \not\precsim_c s_1$ and $s_3 \not\precsim_c s_2$. (Note that all states are equally labeled.) To see $s_1 \precsim_c s_2$ note that $E(s_1) = 0 < 1 = E(s_2)$, and the relation $R = \{ (s_1, s_2) \}$ with the weight function $\Delta(\perp, w) = 1$ will do. $s_2 \not\precsim_c s_1$ as $E(s_2) \not\leqslant E(s_1)$. We have $s_2 \precsim_c s_3$ but $s_3 \not\precsim_c s_2$ as $w \precsim_c s_3$ but $s_3 \not\precsim_c w$. ■

Proposition 25. For any CTMC $\mathcal{C}$:

1. *$s_1 \sim_c s_2$ implies $s_1 \precsim_c s_2$, for any state $s_1, s_2 \in S$.*
2. *$s_1 \precsim_c s_2$ implies $s_1 \precsim_d s_2$ in $\mathrm{emb}(\mathcal{C})$, for any state $s_1, s_2 \in S$.*
3. *$\precsim_c \cap \precsim_c^{-1}$ coincides with $\sim_c$.*
4. *If $\mathcal{C}$ is uniformized then $\precsim_c$ is symmetric and coincides with $\sim_c$.*

Proof:

1. Similar to the proof of Prop. 18.

2. Straightforward.

3. Given the first part of this proposition, it remains to show that strong simulation equivalence contains $\sim_c$. This is done by showing that $\precsim_c \cap \precsim_c^{-1}$ is a strong bisimulation. The labeling condition is obviously fulfilled. Suppose s_1 and s_2 are strongly simulation equivalent. By the same arguments as in the proof of Prop. 21, it follows $\mathbf{P}(s_1, C) = \mathbf{P}(s_2, C)$ for any strong simulation equivalence class C. By the rate condition for $\precsim_c$, we obtain that $E(s_1) = E(s_2)$. Thus,

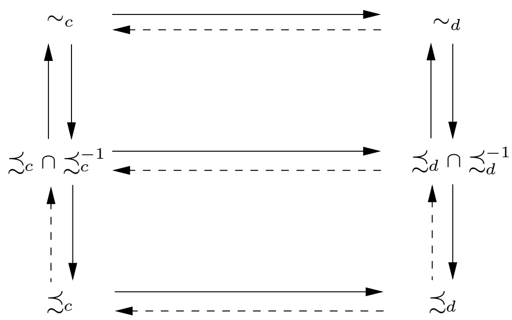
$$\mathbf{R}(s_1, C) \;=\; E(s_1) \cdot \mathbf{P}(s_1, C) \;=\; E(s_2) \cdot \mathbf{P}(s_2, C) \;=\; \mathbf{R}(s_2, C)$$

for all strong simulation equivalence classes C.

4. Follows by straightforward verification from Prop. 18.2.

■

Summarizing the results presented so far yields the two-dimensional spectrum of strong relations on Markov chains depicted below.



$R \longrightarrow R'$ means that R is coarser than R'. The dashed arrows in the continuous setting refer to uniformized CTMCs, i.e., if there is a dashed arrow from R to R', R contains R' for uniformized CTMCs. In the discrete-time setting the dashed arrows refer to DTMCs without absorbing states. Arrows connecting the continuous setting (on the left) with the discrete setting (on the right) relate CTMCs and their embedded DTMCs. Note that for uniformized CTMC $\mathcal{C}$ we have that $emb(\mathcal{C})$ is a DTMC without absorbing states (except for the pathological case where all exit rates in the $\mathcal{C}$ equal zero, in which case all depicted relations agree).

3.3. Weak bisimulation

We consider weak bisimulation which relies on branching bisimulation in the style of van Glabbeek and Weijland [32]. Note that this is not a restriction: whereas for labeled transition systems branching bisimulation is strictly finer than Milner's observational equivalence, they agree for FPSs [9], and thus for DTMCs.

3.3.1. The discrete-time setting

Branching bisimulation [32] only abstracts from stutter-steps inside the equivalence classes, i.e., the only observable moves are those that change the equivalence class. For the probabilistic case this works as follows. Let $\mathcal{D} = (S, \mathbf{P}, L)$ be a DTMC and $R \subseteq S \times S$ an equivalence relation. Any transition from s to s' (i.e., $\mathbf{P}(s, s') > 0$) where s and s' are

R-equivalent is considered an R-*silent* move. Let Silent_R denote the set of states $s \in S$ for which $\mathbf{P}(s, [s]_R) = 1$, i.e., all stochastic states that do not have a successor state outside their R-equivalence class. These states thus can only perform R-silent moves. Stochastic states outside Silent_R thus may leave their R-equivalence class with positive probability by a single transition. Note, in particular, if s is a sub-stochastic state with $\mathsf{Post}(s) = \{ s \}$ or an absorbing state s (i.e., $\mathsf{Post}(s) = \{ s \}$) then s does not belong to Silent_R. For any state $s \notin \mathsf{Silent}_R$, $C \subseteq S$ with $C \cap [s]_R = \varnothing$:

$$\frac{\mathbf{P}(s, C)}{1 - \mathbf{P}(s, [s]_R)}$$

denotes the conditional probability to move from s to some state in C (which is outside $[s]_R$) via a single transition under the condition that from s *no* transition inside $[s]_R$ is taken.

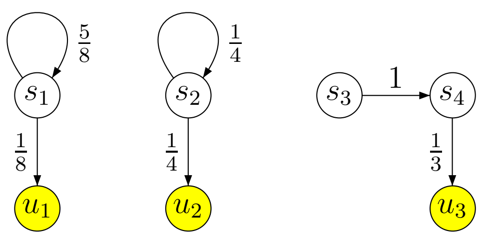
Definition 26. [9] Let $\mathcal{D} = (S, \mathbf{P}, L)$ be an FPS and R an equivalence relation on S. R is a *weak bisimulation* on $\mathcal{D}$ if for all $s_1 \, R \, s_2$:

1. $L(s_1) = L(s_2)$
2. If $\mathbf{P}(s_i, [s_i]_R) < 1$ for $i{=}1, 2$ then for all $C \in S/R$, $C \neq [s_1]_R = [s_2]_R$:
$$\frac{\mathbf{P}(s_1, C)}{1 - \mathbf{P}(s_1, [s_1]_R)} = \frac{\mathbf{P}(s_2, C)}{1 - \mathbf{P}(s_2, [s_2]_R)}$$
3. s_1 can reach a state outside $[s_1]_R$ iff s_2 can reach a state outside $[s_2]_R$.

s_1 and s_2 in $\mathcal{D}$ are weakly bisimilar, denoted $s_1 \approx_d s_2$, iff there exists a weak bisimulation R on $\mathcal{D}$ such that $s_1 \, R \, s_2$. ■

Weakly bisimilar states are equally labeled and their conditional probability to move to another equivalence class (given that they do not stay in their own equivalence class) coincide. Furthermore, by the third condition, for any R-equivalence class C, either all states in C are R-silent (i.e., $\mathbf{P}(s, C) = 1$ for all $s \in C$) or for all $s \in C$ there is a sequence of states $s = s_0, s_1, \ldots, s_n$ with $\mathbf{P}(s_i, s_{i+1}) > 0$ that ends in an equivalence class that differs from C (i.e., $s_n \notin C$).

Example 27.



Consider the FPS depicted above. The equivalence relation R with

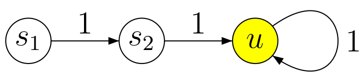
$$S/R = \{ \{s_1, s_2, s_3, s_4\}, \{u_1, u_2, u_3\} \}$$

is a weak bisimulation. This can be seen as follows. For $C = \{ u_1, u_2, u_3 \}$ *and* $s_1, s_2, s_4 \notin \mathsf{Silent}_R$ *we have:*

$$\frac{\mathbf{P}(s_1, C)}{1 - \mathbf{P}(s_1, [s_1])} = \frac{1/8}{1{-}5/8} = \frac{1/4}{1{-}1/4} = \frac{\mathbf{P}(s_2, C)}{1 - \mathbf{P}(s_2, [s_2])} = \frac{1/3}{1} = \frac{\mathbf{P}(s_4, C)}{1 - \mathbf{P}(s_4, [s_4])}$$

Note that $s_3 \in \mathsf{Silent}_R$*. Since* s_3 *can reach a state outside* $[s_3]$ *as* s_1, s_2 *and* s_4*, it follows that* $s_1 \approx_d s_2 \approx_d s_3 \approx_d s_4$*.* ■

Example 28. For the following DTMC, the reachability condition is needed to distinguish states s_1 *and* s_2 *of the following picture from absorbing states with the same label.*



It is not difficult to establish $s_1 \approx_d s_2$*. Note that* s_1 *is* $\approx_d$*-silent while* s_2 *is not. The reachability condition for* s_1 *and* s_2 *is obviously fulfilled. This condition is essential to establish* $s_1 \approx_d s_2$ *and cannot be dropped: otherwise* s_1 *and* s_2 *would be weakly bisimilar to an equally labeled absorbing state.* ■

3.3.2. The continuous-time setting

The intuition behind weak bisimulation on CTMCs is that the time-abstract behaviour of equivalent states is weakly bisimilar (in the sense of the first two conditions of $\approx_d$), and that the "relative speed" of these states to move to another equivalence class is equal. The following result shows that this formulation can be simplified considerably.

Proposition 29. Let $\mathcal{C} = (S, \mathbf{R}, L)$ *be a CTMC and* R *an equivalence relation on* S *with* $s_1 \, R \, s_2$*. The statements 1 and 2 are equivalent:*

1 *If* $s_1,\ s_2 \notin \mathsf{Silent}_R$ *then for all* $C \in S/R$*,* $C \neq [s_1]_R = [s_2]_R$*:*

$$\frac{\mathbf{P}(s_1, C)}{1 - \mathbf{P}(s_1, [s_1]_R)} = \frac{\mathbf{P}(s_2, C)}{1 - \mathbf{P}(s_2, [s_2]_R)} \quad \text{and} \quad \mathbf{R}(s_1, S \setminus [s_1]_R) = \mathbf{R}(s_2, S \setminus [s_2]_R)$$

2 $\mathbf{R}(s_1, C) = \mathbf{R}(s_2, C)$ *for all* $C \in S/R$ *with* $C \neq [s_1]_R = [s_2]_R$*.*

Proof: By showing implication in both directions.

1. Assume that R is an equivalence relation satisfying condition 1. Let $s_1 \, R \, s_2$ and $B =$

$[s_1]_R = [s_2]_R$. We derive:

$$\begin{aligned}
\mathbf{R}(s_1, C) &= E(s_1)\cdot\mathbf{P}(s_1, C) \\
&= \frac{E(s_1)\cdot\mathbf{P}(s_1, C)\cdot\mathbf{P}(s_1, S \setminus B)}{\mathbf{P}(s_1, S \setminus B)} \\
&\overset{1}{=} \frac{E(s_1)\cdot\mathbf{P}(s_2, C)\cdot\mathbf{P}(s_1, S \setminus B)}{\mathbf{P}(s_2, S \setminus B)} \\
&\overset{\text{def. } R}{=} \frac{\mathbf{R}(s_1, S \setminus B)\cdot\mathbf{P}(s_2, C)}{\mathbf{P}(s_2, S \setminus B)} \\
&\overset{1}{=} \frac{\mathbf{R}(s_2, S \setminus B)\cdot\mathbf{P}(s_2, C)}{\mathbf{P}(s_2, S \setminus B)} \\
&= \frac{E(s_2)\cdot\mathbf{P}(s_2, S \setminus B)\cdot\mathbf{P}(s_2, C)}{\mathbf{P}(s_2, S \setminus B)} \\
&= \mathbf{R}(s_2, C)
\end{aligned}$$

We conclude that R is an equivalence relation satisfying condition 2.

2. Assume that R is an equivalence relation satisfying condition 2. Let $s_1 \, R \, s_2$ and $B = [s_1]_R = [s_2]_R$. As R satisfies condition 2 and $s_1 R s_2$, $\mathbf{R}(s_1, C) = \mathbf{R}(s_2, C)$ for all $C \in S/R$ with $C \neq B$. Hence,

$$\mathbf{R}(s_1, S \setminus B) = \sum_{C \in S/R, C \neq B} \mathbf{R}(s_1, C) = \sum_{C \in S/R, C \neq B} \mathbf{R}(s_2, C) = \mathbf{R}(s_2, S \setminus B)$$

and, in particular also $E(s_1) - \mathbf{R}(s_1, B) = E(s_2) - \mathbf{R}(s_2, B)$ (*). If neither s_1 nor s_2 is R-silent, i.e., $\mathbf{P}(s_i, B) < 1$, for i=1, 2, we derive for any $C \in S/R$ with $C \neq B$:

$$\begin{aligned}
\frac{\mathbf{P}(s_1, C)}{1 - \mathbf{P}(s_1, B)} &= \frac{E(s_1) \cdot \mathbf{P}(s_1, C)}{E(s_1) - E(s_1) \cdot \mathbf{P}(s_1, B)} \overset{def.\mathbf{R}}{=} \frac{\mathbf{R}(s_1, C)}{E(s_1) - \mathbf{R}(s_1, B)} \\
&\overset{(*),2}{=} \frac{\mathbf{R}(s_2, C)}{E(s_2) - \mathbf{R}(s_2, B)} = \frac{\mathbf{P}(s_2, C)}{1 - \mathbf{P}(s_2, B)}
\end{aligned}$$

We conclude that R is an equivalence relation satisfying condition 1. ■

This result justifies the following definition of weak bisimulation on CTMCs.

Definition 30. **[17]** Let $\mathcal{C} = (S, \mathbf{R}, L)$ be a CTMC and R an equivalence relation on S. R is a *weak bisimulation* on $\mathcal{C}$ if for all $s_1 \, R \, s_2$:

$$L(s_1) = L(s_2) \quad \text{and} \quad \mathbf{R}(s_1, C) = \mathbf{R}(s_2, C) \text{ for all } C \text{ in } S/R \text{ with } C \neq [s_1]_R.$$

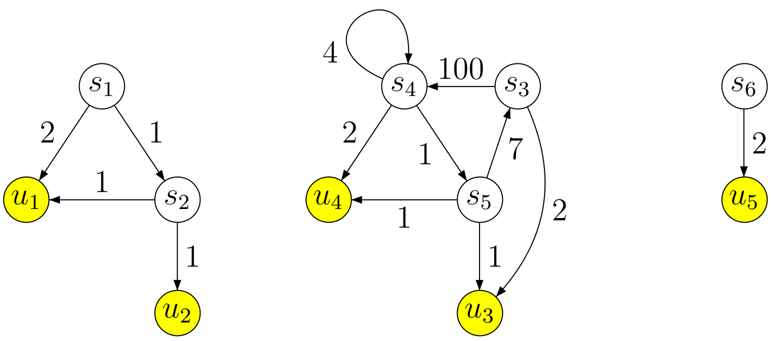
s_1 and s_2 in $\mathcal{C}$ are weakly bisimilar, denoted $s_1 \approx_c s_2$, iff there exists a weak bisimulation R on $\mathcal{C}$ such that $s_1 \, R \, s_2$. ■

Corollary 31. *For CTMC $\mathcal{C}$ with $s_1, s_2 \in S$:*

$$s_1 \approx_c s_2 \text{ implies } s_1 \approx_d s_2 \text{ in } emb(\mathcal{C}).$$

Proof: Follows directly from Prop. 29. ■

Example 32.



The equivalence relation R with

$$S/R \;=\; \{\ \{s_1, s_2, s_3, s_4, s_5, s_6\},\ \{u_1, u_2, u_3, u_4, u_5\}\ \}$$

is a weak bisimulation on the CTMC depicted above. This can be seen as follows. For $C = \{ u_1, u_2, u_3, u_4, u_5 \}$, we have that all s-states enter C with rate 2. Note that the rates between the s-states are not relevant. ■

Proposition 33. For any CTMC $\mathcal{C}$:

1. *$\sim_c$ is strictly finer than $\approx_c$.*

2. *If $\mathcal{C}$ is uniformized then $\approx_c$ coincides with $\sim_c$.*

3. *$\approx_c$ coincides with $\approx_c$ in unif$(\mathcal{C})$.*

Proof:

1. This follows directly from the definitions of $\sim_c$ and $\approx_c$.

2. For weak bisimulation relation R and $s_1 R s_2$ we have $\mathbf{R}(s_1, C) = \mathbf{R}(s_2, C)$ for all $C \in S/R$, $C \neq [s_1]_R = [s_2]_R$. As the CTMC is uniformized, $E(s_1) = E(s_2)$. From these facts it directly follows that $\mathbf{R}(s_1, [s_1]_R) = \mathbf{R}(s_2, [s_1]_R)$, and thus $s_1 \sim_c s_2$.

3. Follows directly from the fact that CTMC $\mathcal{C}$ and *unif*$(\mathcal{C})$ only differ in the rates from a state to itself.

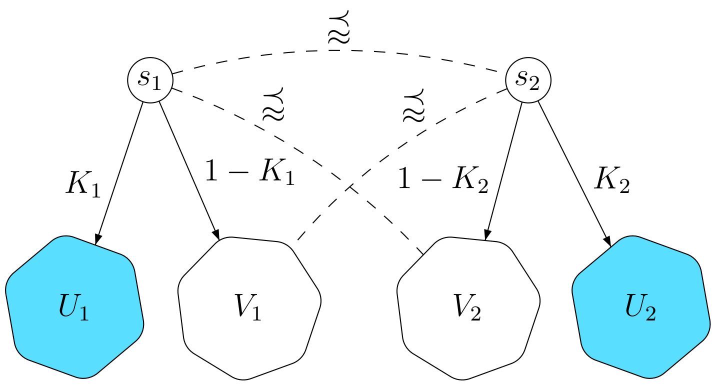
■

The last result can be strengthened as follows. Any state s in $\mathcal{C}$ is weakly bisimilar to s considered as a state in $\mathit{unif}(\mathcal{C})$. (For this, consider the disjoint union of $\mathcal{C}$ and $\mathit{unif}(\mathcal{C})$ as a single CTMC.)

Remark. Prop. 11.2 states that for a uniformized CTMC, $\sim_c$ coincides with $\sim_d$ on the embedded DTMC. The analogue for $\approx_c$ does not hold, as, e.g., in the uniformized CTMC of Example 28 we have $s_1 \approx_d s_2$ but $s_1 \not\approx_c s_2$ as $\mathbf{R}(s_1,[u]) \neq \mathbf{R}(s_2,[u])$. Intuitively, although s_1 and s_2 have the same time-abstract behaviour (up to stuttering) they have distinct timing behaviour. s_1 is "slower than" s_2 as it has to perform a stutter step prior to an observable step (from s_2 to u) while s_2 can immediately perform the latter step. Note that by Prop. 33.2 and Prop. 11.2, $\approx_c$ coincides with $\sim_d$ for uniformized CTMCs. ■

3.4. Weak simulation

In this subsection, we define notions of weak simulation (denoted $\precapprox$) for CTMCs and FPSs that can be considered as "one-sided" weak bisimulations. Roughly speaking, $s_1 \precapprox s_2$ if the successor states of s_1 and s_2 can be grouped into subsets U_i and V_i, for $i{=}1,2$ (assume, for simplicity, $U_i \cap V_i = \varnothing$). All transitions from s_i to V_i are viewed as stutter-steps, i.e., internal transitions that do not change the labeling and that respect $\precapprox$. To that end, any state in V_1 is required to be simulated by s_2 and, symmetrically, any state in V_2 simulates s_1. Transitions from s_i to U_i are regarded as visible steps. Accordingly, we require that the distributions for the conditional probabilities $u_1 \mapsto \mathbf{P}(s_1,u_1)/K_1$ and $u_2 \mapsto \mathbf{P}(s_2,u_2)/K_2$ to move from s_i to U_i are related via a weight function (as for $\precsim_d$). K_i denotes the total probability to move from s_i to a state in U_i – the states that are not simulated by the other – in a single step. The following picture shows the situation for FPSs where in state s_i ($i{=}1,2$) a transition to some state in V_i is made with probability $1{-}K_i$. Note the correspondence with $\approx_d$ (cf. Def. 26), where $[s_1]_R$ plays the role of V_1, while the successors outside $[s_1]_R$ play the role of U_1, and the same for s_2, V_2 and U_2.



For FPSs with sub-stochastic states, we have to consider the deadlock probabilities. This is done as for the strong simulation relations where $\perp$ was treated as a state which is simulated by any other state. For technical reasons we allow $\perp \in U_i$ and $\perp \in V_i$. The possibility of deadlock justifies the need for a reachability condition as for $\approx_d$ (cf. condition 3 of Def. 26). In the continuous setting later on, we deal with a stronger requirement than the reachability condition and require that s_2 is faster than s_1 in the sense that the total rate for s_2 to move to a U_2-state is at least the total rate for s_1 to move to a U_1-state.

3.4.1. The discrete-time setting

We start by defining weak simulation for FPSs. At first reading, consider δ_i as the characteristic function of U_i, and hence, $U_i \cap V_i = \varnothing$. Later on we explain why in fact we need to be more liberal allow for the fragmentation of states, i.e., states that partly belong to U_i and partly to V_i.

Definition 34. Let $\mathcal{D} = (S, \mathbf{P}, L)$ be a FPS and $R \subseteq S \times S$. R is a *weak simulation* on $\mathcal{D}$ iff for $s_1 \, R \, s_2$: $L(s_1) = L(s_2)$ and there exist functions $\delta_i : S_\perp \to [0,1]$ and sets $U_i, V_i \subseteq S_\perp$ (i=1, 2) with

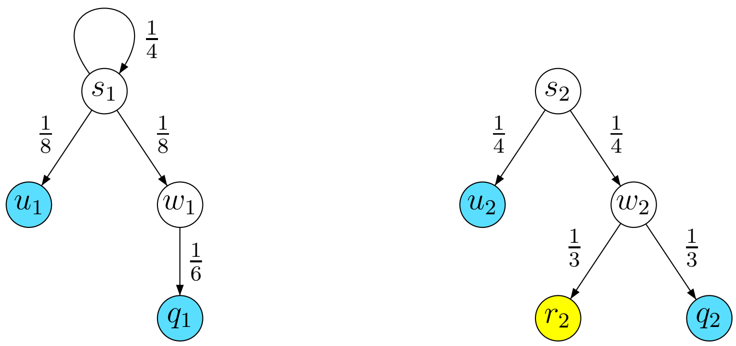
$$U_i \;=\; \{\, u_i \in \mathsf{Post}_\perp(s_i) \mid \delta_i(u_i) > 0 \,\} \text{ and } V_i \;=\; \{\, v_i \in \mathsf{Post}_\perp(s_i) \mid \delta_i(v_i) < 1 \,\}$$

such that:

1. (a) $v_1 \, R \, s_2$ for all $v_1 \in V_1 \setminus \{\perp\}$, and (b) $s_1 \, R \, v_2$ for all $v_2 \in V_2 \setminus \{\perp\}$
2. there exists a function $\Delta : S_\perp \times S_\perp \to [0,1]$ such that:
   (a) $\Delta(u_1, u_2) > 0$ implies $u_1 \in U_1$, $u_2 \in U_2$ and either $u_1 \, R \, u_2$ or $u_1 = \perp$,
   (b) if $K_1 > 0$ and $K_2 > 0$ then for all states $w \in S_\perp$:
   $$K_1 \cdot \sum_{u_2 \in U_2} \Delta(w, u_2) = \delta_1(w) \cdot \mathbf{P}(s_1, w), \quad K_2 \cdot \sum_{u_1 \in U_1} \Delta(u_1, w) = \delta_2(w) \cdot \mathbf{P}(s_2, w)$$
   where $K_i = \sum_{u_i \in U_i} \delta_i(u_i) \cdot \mathbf{P}(s_i, u_i)$ for i=1, 2
3. for $u_1 \in U_1 \setminus \{\perp\}$ there exists a path fragment[2] $s_2, w_1, \ldots, w_n, u_2$ such that $n \geqslant 0$, $s_1 \, R \, w_j$, $0 < j \leqslant n$, and $u_1 \, R \, u_2$.

s_2 weakly simulates s_1 in $\mathcal{D}$, denoted $s_1 \precsim_d s_2$, iff there exists a weak simulation R on $\mathcal{D}$ such that $s_1 \, R \, s_2$. ■

Example 35. In the following FPS we have $s_1 \precsim_d s_2$:



[2]For a formal definition of a path fragment, see page 33.

First, observe that $w_1 \precapprox_d w_2$ *since* $R = \{ (q_1, q_2), (w_1, w_2) \}$ *is a weak simulation, as we may deal with*

- δ_1*, the characteristic function of* $U_1 = \{ q_1, \bot \}$ *(and, thus,* $V_1 = \varnothing$ *and* $K_1 = 1$*)*
- δ_2*, the characteristic function of* $U_2 = \{ r_2, q_2, \bot \}$ *(and* $V_2 = \varnothing$ *and* $K_2 = 1$*)*

and the weight function $\Delta(q_1, q_2) = \Delta(\bot, q_2) = \frac{1}{6}, \Delta(\bot, r_2) = \Delta(\bot, \bot) = \frac{1}{3}$.

To establish a weak simulation for (s_1, s_2) *consider the relation:*

$$R \;=\; \{ (s_1, s_2),\; (u_1, u_2),\; (w_1, w_2),\; (q_1, q_2) \}$$

and put $V_1 = \{ \bot, s_1 \}$ *and* $V_2 = \varnothing$ *while* $U_i = \{ u_i, w_i, \bot \}$ *where* $\delta_1(\bot) = 1/2$, $\delta_i(u_i) = \delta_i(w_i) = \delta_2(\bot) = 1$. *Then,* $K_1 = \frac{1}{8} + \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}, \quad K_2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$. *This yields the following conditional probabilities* $\delta_i(\cdot) \cdot \mathbf{P}(s_i, \cdot)/K_i$ *for the* U*-successors of* s_1 *and* s_2*:*

$$u_1 : \tfrac{1}{4}, \quad w_1 : \tfrac{1}{4}, \quad \bot : \tfrac{1}{2}, \quad u_2 : \tfrac{1}{4}, \quad w_2 : \tfrac{1}{4}, \quad \textit{and} \quad \bot : \tfrac{1}{2}.$$

Note that, e.g., $\frac{\delta_1(u_1) \cdot \mathbf{P}(s_1, u_1)}{K_1} = \frac{1}{4}$ *and* $\frac{\delta_1(\bot) \cdot \mathbf{P}(s_1, \bot)}{K_1} = \frac{1}{2}$. *Hence, an appropriate weight function is:* $\Delta(u_1, u_2) = \Delta(w_1, w_2) = \frac{1}{4}$, $\Delta(\bot, \bot) = \frac{1}{2}$, *and* $\Delta(\cdot) = 0$ *for the remaining cases. Thus, according to Def. 34,* R *is a weak simulation, and as* $s_1 \, R \, s_2$*, it follows* $s_1 \precapprox_d s_2$. ∎

Remark. Def. 34 allows the case $U_1 = \varnothing$ (i.e., K_1=0) or symmetrically $U_2 = \varnothing$ (i.e., K_2=0). A few remarks on these special cases are in order.

- If $U_1 = \varnothing$ then $\mathsf{Post}(s_1) \subseteq s_2{\downarrow}_{\precapprox_d}$, i.e., all successors of s_1 are weakly simulated by s_2. In this case, no further requirements are made (i.e., condition 3 of Def. 34 is vacuously true). For condition 2 we may put $\Delta(\bot, u_2) = \mathbf{P}(s_2, u_2)$ for all $u_2 \in S_\bot$. In particular, any state s_1 with $\mathsf{Post}(s_1) \subseteq \{ s_1 \}$ is weakly simulated by any equally labeled state s_2. This corresponds to the view that the self-loop $s_1 \to s_1$ is invisible, i.e., a stutter step.

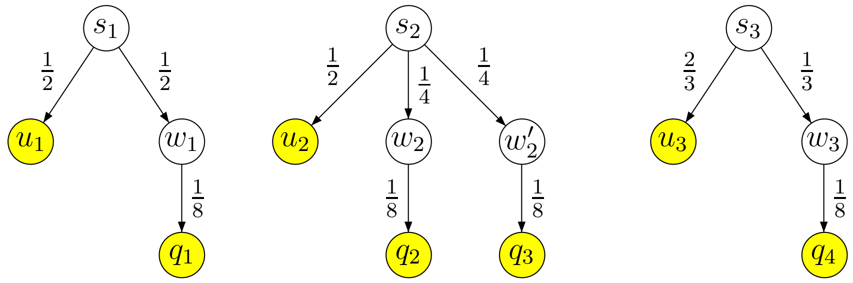
- Note that the reachability condition is redundant when $U_2 \neq \varnothing$. If $U_1 = \varnothing$ then this condition holds. Otherwise, if $K_1 > 0$ and $K_2 > 0$, the weight function conditions (cf. condition 2 of Def. 34) ensure that any visible transition $s_1 \to u_1 \in U_1$ is matched by a visible transition $s_2 \to u_2 \in U_2$ where $\Delta(u_1, u_2) > 0$ (and hence, $u_1 R u_2$).

- If $U_2 = \varnothing$ and $U_1 \neq \varnothing$ then the reachability condition (cf. condition 3 of Def. 34) ensures that for any visible step $s_1 \to u_1$ (with $u_1 \in U_1$), s_2 can reach a state u_2 that simulates u_1 via a path fragment through states that simulate s_1. The intuition behind this condition is that s_1 is able to perform a visible move which has to be matched by path fragments that start with stutter-steps $s_2 \to w_1 \to \ldots \to w_n$ followed by a move $w_n \to u_2$ which can be viewed as being visible and as mimicking the transition $s_1 \to u_1$.

■

In the previous example, we have used the special case where $\delta_i(s) \in \{0,1\}$ for any state $s \neq \bot$. In this case, δ_i is the characteristic function of U_i, and the sets U_i and V_i are disjoint. In general, though, things are more complicated and we need to construct U_i and V_i using *fragments* of states. That is, we deal with functions δ_i where $0 \leqslant \delta_i(s) \leqslant 1$ for state s. Intuitively, the $\delta_i(s)$-fragment of state s belongs to U_i, while the remaining part (the $(1-\delta_i(s))$-part) of s belongs to V_i. The use of fragments of states is exemplified in the following example.

Example 36. In the following FPS, we have $s_1 \precsim_d s_2$ and $s_2 \precsim_d s_3$.



To establish weak simulations for (s_1, s_2) and (s_2, s_3), we do not need to consider fragments of states. For (s_1, s_2), we can deal with the partitioning $V_1^{1,2} = V_2^{1,2} = \varnothing$, $K_1^{1,2} = K_2^{1,2} = 1$ and

$$\Delta_{1,2}(u_1, u_2) = \frac{1}{2}, \quad \Delta_{1,2}(w_1, w_2) = \Delta_{1,2}(w_1, w_2') = \frac{1}{4}$$

and $\Delta_{1,2}(\cdot) = 0$ otherwise. For (s_2, s_3) we may deal with $V_1^{2,3} = \{w_2'\}$, $V_2^{2,3} = \varnothing$, $K_1^{2,3} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ and $K_2^{2,3} = 1$ and the weight function

$$\Delta_{2,3}(u_2, u_3) \;=\; \frac{2}{3}, \quad \Delta_{2,3}(w_2, w_3) = \frac{1}{3}$$

and $\Delta_{2,3}(\cdot) = 0$ in all other cases.

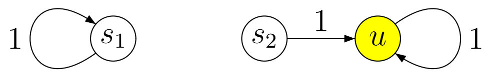
If, however, we do not consider fragments of states, a weak simulation between s_1 and s_3 cannot be established: as $s_1 \not\precsim_d w_3$, $s_3 \to w_3$ cannot be considered a stutter step and, hence, $w_3 \in U_2^{1,3}$ (and $V_2^{1,3} = \varnothing$). For s_1 there are two possible partitionings: (I) $V_1^{1,3} = \varnothing$ and $U_1^{1,3} = \{u_1, w_1\}$, or (II) $V_1^{1,3} = \{w_1\}$ and $U_1^{1,3} = \{u_1\}$. For (I) we obtain the distribution 1/2–1/2 for the dark and white states, while in case (II) we obtain the distribution 1–0 for the "visible" successors of s_1 and the distribution 2/3–1/3 for the white and dark successors of s_3. More precisely, in case (I) we have to deal with $K_1^{1,3} = K_2^{1,3} = 1$ and $\delta_1^{1,3}(\cdot) = 1$, $\delta_2^{1,3}(\cdot) = 1$. But then, there are no weights $\Delta(u_1, u_3)$, $\Delta(w_1, w_3)$ satisfying condition 2.(ii) which would require $\Delta(u_1, u_3) = \mathbf{P}(s_1, u_1) = \frac{1}{2}$, $\Delta(w_1, w_3) = \mathbf{P}(s_1, w_1) = \frac{1}{2}$, and $\Delta(u_1, u_3) = \mathbf{P}(s_1, u_1) = \frac{2}{3}$, $\Delta(w_1, w_3) = \mathbf{P}(s_1, w_1) = \frac{1}{3}$. Similar arguments show the impossibility of case (II). In none of these cases, condition 2. of Def. 34 is satisfied.

By considering fragments *of states (using δ_i), it is possible to "split" w_1 into two fragments: e.g., one half belonging to $V_1^{1,3}$ and the other half to $U_1^{1,3}$, i.e.,*

$$\delta_1^{1,3}(w_1) = \frac{1}{2}, \quad \delta_1^{1,3}(u_1) = 1$$

while $\delta_2^{1,3}(u_3) = \delta_2^{1,3}(w_3) = 1$. *Then,* $K_2^{1,3} = 1$ *and* $K_1^{1,3} = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$. *With the weight function* $\Delta_{1,3}(u_1, u_2) = \frac{2}{3}$ *and* $\Delta_{1,3}(w_1, w_3) = \frac{1}{3}$ *we establish* $s_1 \precapprox_d s_3$. ■

Remark. Due to the reachability condition (condition 3 in Def. 34), $\precapprox_d$ is not symmetric, even for DTMCs without absorbing states. The reason is that the reachability condition is one-sided and treats s_1 and s_2 in a different way.



The above figure illustrates a DTMC without absorbing state where $s_1 \precapprox_d s_2$ but $s_2 \not\precapprox_d s_1$. Recall that $\precsim_d$ coincides with $\sim_d$ for DTMCs without absorbing states. Due to the non-symmetry of $\precapprox_d$ such result cannot be established for $\precapprox_d$. ■

The proof of the next result shows that considering state-fragments is necessary in order to establish the transitivity of $\precapprox_d$.

Proposition 37. $\precapprox_d$ *is a preorder.*

Proof: Reflexivity directly follows from Def. 34. Transitivity is proven as follows. Let $R_{1,2}$ and $R_{2,3}$ be weak simulations on FPS $\mathcal{D} = (S, \mathbf{P}, L)$. We show that:

$$R \;=\; R_{1,2} \circ R_{2,3} \;=\; \{ (s_1, s_3) \mid \exists s_2 \in S.\, (s_1\, R_{1,2}\, s_2 \;\wedge\; s_2\, R_{2,3}\, s_3) \}$$

is a weak simulation. Assume $s_1 R s_3$. Then there exists a state s_2 such that $s_1\, R_{1,2}\, s_2$ and $s_2\, R_{2,3}\, s_3$. We check the conditions of Def. 34 for R. Let $\delta_1^{1,2}$, $\delta_2^{1,2}$, $U_1^{1,2}$, $U_2^{1,2}$, $V_1^{1,2}$, $V_2^{1,2}$, $K_1^{1,2}$, $K_2^{1,2}$, and $\Delta_{1,2}$ be the components as in Def. 34 for establishing $s_1\, R_{1,2}\, s_2$. For the sake of simplicity, we assume that each one-step successor state of s_1 either belongs to $U_1^{1,2}$ or to $V_1^{1,2}$ but *not* to both, i.e., the function $\delta_1^{1,2}$ is the characteristic function of $U_1^{1,2}$.[3] Then, $K_1^{1,2} = \mathbf{P}(s_1, U_1^{1,2})$. The same is assumed for states s_2 and s_3, and we use the notations $U_1^{2,3}$, $U_2^{2,3}$, etc. with the obvious meaning. Let $U_2 \;=\; U_2^{1,2} \cap U_1^{2,3}$ and

$$\begin{aligned} U_1 &= \{ u_1 \in U_1^{1,2} \mid \Delta_{1,2}(u_1, u_2) > 0 \text{ for some } u_2 \in U_2 \}, \\ U_3 &= \{ u_3 \in U_2^{2,3} \mid \Delta_{2,3}(u_2, u_3) > 0 \text{ for some } u_2 \in U_2 \}. \end{aligned}$$

Note that $u_i \in U_i$ implies $\mathbf{P}(s_i, u_i) > 0$ for i=1, 3. For $u_1 \in U_1$ and $u_3 \in U_3$ let:

$$\begin{aligned} \delta_1(u_1) &= \sum_{u_2 \in U_2} \Delta_{1,2}(u_1, u_2) \cdot \frac{K_1^{1,2}}{\mathbf{P}(s_1, u_1)} \\ \delta_3(u_3) &= \sum_{u_2 \in U_2} \Delta_{2,3}(u_2, u_3) \cdot \frac{K_2^{2,3}}{\mathbf{P}(s_3, u_3)} \end{aligned}$$

[3] The justification for this simplification is as follows. For the proof of the general case we have to replace any occurrence of $\mathbf{P}(s_1, u_1)$ for $u_1 \in U_1^{1,2}$ by $\delta_1^{1,2}(u_1) \cdot \mathbf{P}(s_1, u_1)$ and, similarly, $\mathbf{P}(s_1, v_1)$ for $v_1 \in V_1^{1,2}$ by $(1 - \delta_1^{1,2}(v_1)) \cdot \mathbf{P}(s_1, v_1)$.

Let $\delta_1(w) = 0$ if $w \in S \setminus U_1$, $\delta_3(w) = 0$ if $w \in S \setminus U_3$, and:

$$\begin{aligned} K_1 &= \sum_{u_1 \in U_1} \delta_1(u_1) \cdot \mathbf{P}(s_1, u_1) = \sum_{u_1 \in U_1, u_2 \in U_2} \Delta_{1,2}(u_1, u_2) \cdot K_1^{1,2} \\ K_3 &= \sum_{u_3 \in U_3} \delta_3(u_3) \cdot \mathbf{P}(s_3, u_3) = \sum_{u_2 \in U_2, u_3 \in U_3} \Delta_{2,3}(u_2, u_3) \cdot K_2^{2,3} \\ K_2 &= \sum_{u_2 \in U_2} \mathbf{P}(s_2, u_2). \end{aligned}$$

V_1 denotes the set of one-step successors $v_1 \in S_\perp$ of s_1 such that $\delta_1(v_1) < 1$. V_3 has the corresponding meaning for state s_3.

We check the conditions of Def. 34. We first show that $0 < \delta_i(u_i) \leqslant 1$ for all $u_i \in S$. The fact that $\delta_i(u_i) > 0$ is clear. $\delta_i(u_i) \leqslant 1$ follows from:

$$\sum_{u_2 \in U_2} \Delta_{1,2}(u_1, u_2) \cdot K_1^{1,2} \leqslant \sum_{u_2 \in S} \Delta_{1,2}(u_1, u_2) \cdot K_1^{1,2} = \mathbf{P}(s_1, u_1)$$

for any state $u_1 \in U_1$. Note that $K_1^{1,2} \cdot \sum_{u_1 \in U_1} \Delta_{1,2}(u_1, u_2) < \mathbf{P}(s_1, u_1)$ is possible because there might be states $u_2 \in U_2^{1,2} \setminus U_1^{2,3}$. A similar observation holds for $u_3 \in U_3$. We now check the conditions of Def. 34.

1. (a) Let $v_1 \in V_1 \setminus \{\perp\}$. Distinguish two cases: (i) $v_1 \notin U_1^{1,2}$ and (ii) $v_1 \in U_1^{1,2}$. For case (i), $v_1 \in V_1^{1,2}$, and by the fact that $s_1 \precsim_d s_2$, it follows $v_1 \; R_{1,2} \; s_2$. Since $s_2 \, R_{2,3} \, s_3$ it follows from the definition of R that $v_1 \, R \, s_3$. Case (ii): let $v_1 \in V_1 \setminus \{\perp\} \cap U_1^{1,2}$. Note that $v_1 \in V_1$ implies $\delta_1(v_1) < 1$. Hence,

   $$K_1^{1,2} \cdot \sum_{u_2 \in U_2} \Delta_{1,2}(v_1, u_2) < \mathbf{P}(s_1, v_1).$$

   On the other hand,

   $$\begin{aligned} \mathbf{P}(s_1, v_1) &= K_1^{1,2} \cdot \sum_{u_2 \in U_2^{1,2}} \Delta_{1,2}(v_1, u_2) \\ &= K_1^{1,2} \cdot \sum_{u_2 \in U_2} \Delta_{1,2}(v_1, u_2) + K_1^{1,2} \cdot \sum_{u_2 \in U_2^{1,2} \setminus U_2} \Delta_{1,2}(v_1, u_2) \end{aligned}$$

   Hence, there exists $u_2 \in U_2^{1,2} \setminus U_2$ with $\Delta_{1,2}(v_1, u_2) > 0$. Then, $u_2 \in U_2^{1,2} \setminus U_1^{2,3}$ and therefore $u_2 \in V_1^{2,3}$. We directly obtain from the fact that $s_1 \, R \, s_3$ that $v_1 \, R_{1,2} \, u_2 \; R_{2,3} \; s_3$, and, hence, $v_1 \, R \, s_3$.

   (b) In a similar way, we obtain $s_1 \, R \, v_3$ for $v_3 \in V_3 \setminus \{\perp\}$.

2. Assume $U_1, U_3 \neq \varnothing$. Hence, $K_1 > 0$ and $\min\{K_1^{2,3}, K_2^{1,2}\} \geqslant K_2 > 0$. We will define a function Δ such that with the above definitions of $\delta_1, \delta_3, U_1, U_3, V_1, V_3, K_1$, and K_3, conditions 2.(i) and 2.(ii) of Def. 34 are satisfied. We first make the following two observations:

   (a) $K_1 \cdot K_2^{1,2} = K_1^{1,2} \cdot K_2$ and $K_3 \cdot K_1^{2,3} = K_2^{2,3} \cdot K_2$

For the first equation this can be seen as follows:

$$K_1 \cdot K_2^{1,2} \;=\; \sum_{u_1 \in U_1} \sum_{u_2 \in U_2} \Delta_{1,2}(u_1,u_2) \cdot K_1^{1,2} \cdot K_2^{1,2}$$

$$K_1^{1,2} \cdot K_2 \;=\; K_1^{1,2} \cdot \sum_{u_2 \in U_2} \mathbf{P}(s_2,u_2) \;=\; K_1^{1,2} \cdot \sum_{u_2 \in U_2} \sum_{u_1 \in U_1} \Delta_{1,2}(u_1,u_2) \cdot K_2^{1,2}$$

(b) If $\Delta_{2,3}(u_2,u_3) > 0$ and $u_2 \in U_2$ then $u_3 \in U_3$. Hence, for any state $u_2 \in U_2$:

$$\sum_{u_3 \in U_3} \Delta_{2,3}(u_2,u_3) \;=\; \sum_{u_3 \in S} \Delta_{2,3}(u_2,u_3) \;=\; \mathbf{P}(s_2,u_2)/K_1^{2,3}$$

Similarly, we have for all states $u_2 \in U_2$:

$$\sum_{u_1 \in U_1} \Delta_{1,2}(u_1,u_2) \;=\; \sum_{u_1 \in S} \Delta_{1,2}(u_1,u_2) \;=\; \mathbf{P}(s_2,u_2)/K_2^{1,2}$$

These two observations provide us the means to check condition 2. of Def. 34:

(i) Let $\Delta : U_1 \times U_3 \to [0,1]$ be given by:

$$\Delta(u_1,u_3) \;=\; \sum_{u_2 \in U_2} \Delta_{1,2}(u_1,u_2) \cdot \Delta_{2,3}(u_2,u_3) \cdot \frac{K_2^{1,2} \cdot K_1^{2,3}}{\mathbf{P}(s_2,u_2) \cdot K_2} \qquad (1)$$

If $\Delta(u_1,u_3) > 0$ then there exists some $u_2 \in S$ with

$$\Delta_{1,2}(u_1,u_2) > 0 \text{ and } \Delta_{2,3}(u_2,u_3) > 0.$$

Hence, $u_2 \in U_2$ and $u_1 \, R_{1,2} \, u_2$ and $u_2 \, R_{2,3} \, u_3$, and by definition of R, $u_1 \, R \, u_3$.

(ii) Using the definition of Δ (cf. equation (1)), we derive for state $u_1 \in U_1$:

$$\begin{aligned}
& K_1 \cdot \sum_{u_3 \in U_3} \Delta(u_1,u_3) \\
=\; & K_1 \cdot \sum_{u_3 \in U_3} \sum_{u_2 \in U_2} \Delta_{1,2}(u_1,u_2) \cdot \Delta_{2,3}(u_2,u_3) \cdot \frac{K_2^{1,2} \cdot K_1^{2,3}}{\mathbf{P}(s_2,u_2) \cdot K_2} \\
=\; & K_1 \cdot \sum_{u_2 \in U_2} \Delta_{1,2}(u_1,u_2) \cdot \frac{K_2^{1,2} \cdot K_1^{2,3}}{\mathbf{P}(s_2,u_2) \cdot K_2} \cdot \underbrace{\sum_{u_3 \in U_3} \Delta_{2,3}(u_2,u_3)}_{=\, \mathbf{P}(s_2,u_2)/K_1^{2,3}, \text{ see (b)}} \\
=\; & \underbrace{\frac{K_1 \cdot K_2^{1,2}}{K_2}}_{=\, K_1^{1,2}, \text{ see (a)}} \cdot \sum_{u_2 \in U_2} \Delta_{1,2}(u_1,u_2) \\
=\; & K_1^{1,2} \cdot \sum_{u_2 \in U_2} \Delta_{1,2}(u_1,u_2) \\
=\; & \delta_1(u_1) \cdot \mathbf{P}(s_1,u_1)
\end{aligned}$$

Similarly, we get $K_3 \cdot \sum_{u_1 \in U_1} \Delta(u_1,u_3) \;=\; \delta_3(u_3) \cdot \mathbf{P}(s_3,u_3)$.

3. Let $u_1 \in U_1$. By definition of U_1, there exists $u_2 \in U_2$ such that $\Delta_{1,2}(u_1, u_2) > 0$. Condition 3 of Def. 34 applied to $s_2 \, R_{2,3} \, s_3$ and the successor state $u_2 \in U_1^{2,3}$ implies the existence of a path fragment $s_3, w_1, \ldots, w_n, u_3$ with $n \geqslant 0$ such that $s_2 \, R_{2,3} \, w_j$ (for $0 < j \leqslant n$) and $u_2 \, R_{2,3} \, u_3$. Since $s_1 \, R_{1,2} \, s_2$, we obtain $s_1 \, R \, w_j$ (for $0 < j \leqslant n$). Because $\Delta_{1,2}(u_1, u_2) > 0$, $u_1 \, R_{1,2} \, u_2$ and, hence, by definition of R, $u_1 \, R \, u_3$.

■

Proposition 38. *For any FPS $\mathcal{D}$:*

$$s_1 \approx_d s_2 \text{ implies } s_1 \precapprox_d s_2, \text{ and } s_1 \precsim_d s_2 \text{ implies } s_1 \precapprox_d s_2.$$

Proof: As the second conjunct follows by easy verification (put $V_1 = V_2 = \varnothing$ and let δ_i be the characteristic function of $U_i = \mathsf{Post}_\perp(s_i)$) we concentrate on the proof of the first part. Let $[s] = [s]_{\approx_d}$ and $s_1 \approx_d s_2$ in $\mathcal{D}$ with $B = [s_1] = [s_2]$. We consider U_i and V_i given by δ_i as the characteristic function of the set consisting of all successor states of s_i outside B, i.e., $U_i = \mathsf{Post}_\perp(s_i) \setminus B$, and $V_i = \mathsf{Post}_\perp(s_i) \cap B$. Then, $K_i = 1 - \mathbf{P}(s_i, B)$. By Prop. 15 the existence of a weight function for the distributions

$$u_1 \mapsto \frac{\mathbf{P}(s_1, u_1)}{1 - \mathbf{P}(s_1, B)}, \quad u_2 \mapsto \frac{\mathbf{P}(s_2, u_2)}{1 - \mathbf{P}(s_2, B)}$$

(where $\mathbf{P}(s_i, B) < 1$) for the U_i-states can be established. Distinguish two cases.

- If $\mathbf{P}(s_1, B) = 1$ then $U_1 = \mathsf{Post}_\perp(s_1) \setminus B = \varnothing$ and $K_1 = 0$. Thus, $s_1 \precapprox_d s_2$.
- If $\mathbf{P}(s_1, B) < 1$ and $\mathbf{P}(s_2, B) = 1$ then $K_2 = 0$ and $K_1 > 0$ and by the reachability condition of $\approx_d$, s_2 can reach some $s_2' \in B$ with $\mathbf{P}(s_2', B) < 1$. By the last condition of Def. 34 it follows $s_1 \precapprox_d s_2$.

■

3.4.2. The continuous-time setting

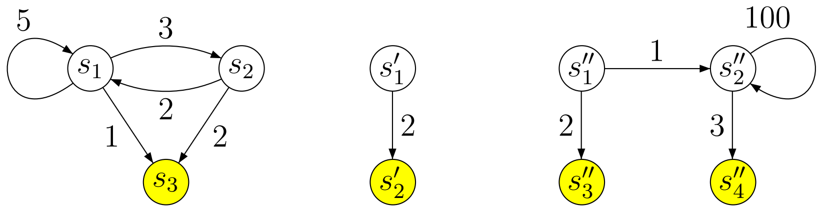
Definition 39. Let $\mathcal{C} = (S, \mathbf{R}, L)$ be a CTMC and $R \subseteq S \times S$. R is a *weak simulation* on $\mathcal{C}$ iff for $s_1 \, R \, s_2$: $L(s_1) = L(s_2)$ and there exist $\delta_i : S \to [0,1]$ and $U_i, V_i \subseteq S$ (i=1, 2) satisfying conditions 1. and 2. of Def. 34 (ignoring $\perp$) and the rate condition:

$$\sum_{u_1 \in U_1} \delta_1(u_1) \cdot \mathbf{R}(s_1, u_1) \;\leqslant\; \sum_{u_2 \in U_2} \delta_2(u_2) \cdot \mathbf{R}(s_2, u_2).$$

s_2 weakly simulates s_1 in $\mathcal{C}$, denoted $s_1 \precapprox_c s_2$, iff there exists a weak simulation R on $\mathcal{C}$ such that $s_1 \, R \, s_2$. ■

The rate condition which replaces the reachability condition in FPSs states that s_2 is "faster than" s_1 in the sense that the total rate to move from s_2 to (the δ_2-part of) the U_2-states is at least the total rate to move from s_1 to (the δ_1-part of) the U_1-states. s_2 can thus carry out visible transitions at least as fast as s_1 can. Note that $K_i \cdot E(s_i) = \sum_{u_i \in U_i} \delta_i(u_i) \cdot \mathbf{R}(s_i, u_i)$. Hence, the rate condition can be rewritten as $K_1 \cdot E(s_1) \leqslant K_2 \cdot E(s_2)$. In particular, $K_2 = 0$ implies $K_1 = 0$. Therefore, a reachability condition as for weak simulation on FPSs is not needed here.

Example 40.



(1) (2) (3)

Consider the three CTMCs depicted above. We have $s_1 \precapprox_c s_1'$*, since there exists a relation*

$$\precapprox \;=\; \{\, (s_1, s_1'), (s_3, s_2'), (s_2', s_3), (s_2, s_1') \,\}$$

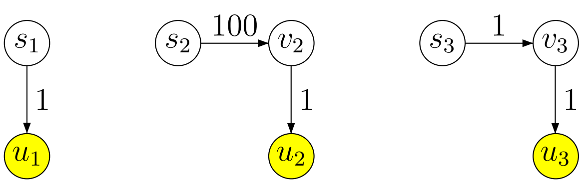
with $U_1 = \{\, s_3 \,\}, V_1 = \{\, s_1, s_2 \,\}$, $\delta_1(s_3) = 1$ *and 0 otherwise,* $U_2 = \{\, s_2' \,\}$, $V_2 = \varnothing$, $\delta_2(s_2') = 1$ *and 0 otherwise, and* $\Delta(s_3, s_2') = \Delta(s_2', s_3) = 1$ *and 0 otherwise. It follows that* $K_1 = \frac{1}{9}$ *and* $K_2 = 1$*. It is not difficult to check that indeed all constraints of Def. 39 are fulfilled, e.g., for the rate condition we obtain* $\frac{1}{9}{\cdot}9 \leqslant 1{\cdot}2$*. Note that* $s_1 \not\precapprox_c s_2$ *if* $\mathbf{R}(s_2, s_3) > 2$ *(rather than being equal to 2), since then* $s_2 \precapprox s_1'$ *can no longer be established.*

We further have $s_1' \precapprox_c s_1''$ *since there exists a relation*

$$\precapprox \;=\; \{\, (s_1', s_1''), (s_1', s_2''), (s_2', s_3''), (s_3'', s_2'), (s_2', s_4''), (s_4'', s_2') \,\}$$

with $U_1 = \{\, s_2' \,\}$, $V_1 = \varnothing$, $K_1 = 1$, *and* $\delta_1(s_2') = 1$ *and 0 otherwise,* $U_2 = \{\, s_3'' \,\}$, $V_2 = \{\, s_2'' \,\}$, $\delta_2(s_3'') = 1$ *and 0 otherwise,* $K_2 = \frac{2}{3}$ *and* $\Delta(s_3'', s_2') = \Delta(s_2', s_3'') = 1$*. It is straightforward to check that indeed all constraints of Def. 39 are fulfilled.*

Example 41. The following figure illustrates a CTMC where $s_3 \precapprox_c s_2 \precapprox_c s_1$ *while* $s_1 \not\precapprox_c s_2$ *and* $s_2 \not\precapprox_c s_3$*.*



The relation $R_{2,1} = \{\, (s_2, s_1), (v_2, s_1), (u_2, u_1) \,\}$ *is a weak simulation as* $s_2 \rightarrow v_2$ *can be viewed as a stutter step. The fact that* $s_3 \precapprox_c s_2$ *follows from* $R_{3,2} = \{\, (s_3, s_2), (v_3, v_2), (u_3, u_2) \,\}$ *being a weak (and even strong) simulation. As* s_2 *is slower than* s_1 *and* s_3 *is slower than* s_2*, intuitively,* $s_1 \not\precapprox_c s_2$ *and* $s_2 \not\precapprox_c s_3$*. That this indeed is the case can be seen as follows.*

(1) *For* (s_1, s_2)*, a weak simulation cannot be established as (due to the labeling condition) the only possibility would be to let* $v_2 \in V_2$ *and* $u_1 \in U_1$*. But then, the rate condition would be violated as* $K_1{\cdot}E(s_1) = 1 > 0 = K_2{\cdot}E(s_2)$*.*

(2) *To see why* $s_2 \not\precapprox_c s_3$*, assume that there is a weak simulation* R *containing* (s_2, s_3)*. As in (1),* $v_2 \not\precapprox_c s_3$ *and hence,* $(v_2, s_3) \notin R$*, i.e.,* v_2 *cannot be put into* V_1 *and we have to deal with* $\delta_1(v_2) = 1$, $U_1 = \{\, v_2 \,\}$ *and* $K_1 = 1$*. But then, the rate condition is invalidated:* $K_1 \cdot E(s_2) = 1 \cdot 100 \;\not\leqslant\; K_2 \cdot E(s_3) = K_2 \in [0, 1]$*.*

■

Remark. If one of the states s_1 or s_2 with $s_1 \precapprox_c s_2$ is absorbing, a simplified characterization of $\precapprox_c$ can be obtained.

1. If s_1 is absorbing then $s_1 \precapprox_c s_2$ if and only if $L(s_1) = L(s_2)$. The implication from right to left immediately follows from the labeling condition (cf. condition 1 in Def. 39). For the other direction, the choices $U_1 = V_1 = \varnothing$, $K_1 = 0$, $U_2 = \mathsf{Post}(s_2)$, and $V_2 = \varnothing$ fulfill the conditions of Def. 39.

2. If s_2 is absorbing then $s_1 \precapprox_c s_2$ if and only if all states (including s_1) reachable from s_1 have the same labeling as s_2. The "only if" part can be seen as follows. When s_2 is absorbing and $s_1 \precapprox_c s_2$ then $U_2 = \varnothing$. By the rate condition, we obtain that $K_1 \cdot E(s_1) \leqslant K_2 \cdot E(s_2) = 0$. Thus, $K_1 = 0$ or $E(s_1) = 0$. If $E(s_1) = 0$ then s_1 is absorbing and the claim is obvious as s_1 is the only state reachable from s_1 and $L(s_1) = L(s_2)$. If $E(s_1) > 0$ and $K_1 = 0$ then $U_1 = \varnothing$ and

$$\mathsf{Post}(s_1) \;=\; V_1 \;\subseteq\; s_2{\downarrow}_R \;\subseteq\; \{s' \in S \mid L(s') = L(s_2)\}.$$

   All states reachable from s_1 thus have the same labeling as s_2.

   Vice versa, if s_2 is absorbing and $L(s') = L(s_2)$ for any state s' reachable from s_1 then the relation R consisting of all pairs (s', s_2) is a weak simulation.

■

Proposition 42. $\precapprox_c$ *is a preorder.* *Proof:* The proof is the same as that of Prop. 37, except that we have to check the rate condition instead of the reachability condition. Using the notations as in the proof of Prop. 37, we have:

$$\begin{aligned}
K_2 \cdot E(s_2) &= \sum_{u_2 \in U_2} \mathbf{P}(s_2, u_2) \cdot E(s_2) \\
&= \sum_{u_2 \in U_2} \sum_{u_3 \in S} \Delta_{2,3}(u_2, u_3) \cdot K_1^{2,3} \cdot E(s_2) \\
&= \sum_{u_2 \in U_2} \sum_{u_3 \in U_3} \Delta_{2,3}(u_2, u_3) \cdot \underbrace{K_1^{2,3} \cdot E(s_2)}_{\leqslant K_2^{2,3} \cdot E(s_3)} \\
&\leqslant \sum_{u_2 \in U_2} \sum_{u_3 \in U_3} \Delta_{2,3}(u_2, u_3) \cdot K_2^{2,3} \cdot E(s_3) \\
&= K_3 \cdot E(s_3)
\end{aligned}$$

With the same arguments, we can show that $K_1 \cdot E(s_1) \leqslant K_2 \cdot E(s_2)$. This yields

$$K_1 \cdot E(s_1) \;\leqslant\; K_2 \cdot E(s_2) \;\leqslant\; K_3 \cdot E(s_3).$$

■

Proposition 43. For CTMC $\mathcal{C}$ and states $s_1, s_2 \in S$:

1. *$s_1 \precapprox_c s_2$ implies $s_1 \precapprox_d s_2$ in emb($\mathcal{C}$).*
2. *$s_1 \approx_c s_2$ implies $s_1 \precapprox_c s_2$.*
3. *$\precapprox_c$ coincides with $\precapprox_c$ in unif($\mathcal{C}$).*

Proof:

1. Easy verification.
2. Using Prop. 31, this proof goes along similar lines as the proof of $\approx_d \subseteq \precapprox_d$.
3. ($\Rightarrow$) Let $s_1 \precapprox_c s_2$ in $\mathcal{C}$ and let R, δ_i, U_i, V_i, K_i (for i=1, 2) and Δ as in Def. 39. The same components U_i, V_i and Δ can be used to show that R is a weak simulation on *unif*($\mathcal{C}$) $= (S, \overline{\mathbf{R}}, L)$. Let $\overline{\delta_1}(s) = \delta_1(s)$ if $s \neq s_1$ and

$$\overline{\delta_1}(s_1) = \delta_1(s_1) \cdot \frac{\mathbf{R}(s_1, s_1)}{\overline{\mathbf{R}}(s_1, s_1)},$$

and $\overline{\delta_2}$ be defined similarly. We show that R is a weak simulation on *unif*($\mathcal{C}$) by checking the conditions of Def. 39. It suffices to check conditions 2.(ii) and the rate condition; the other constraints are clear.

2.(ii). Let

$$\overline{K_i} \;=\; \sum_{u_i \in U_i} \overline{\delta_i}(u_i) \cdot \overline{\mathbf{P}}(s_i, u_i)$$

where $\overline{\mathbf{P}}(s_i, u_i) = \overline{\mathbf{R}}(s_i, u_i)/E$ are the transition probabilities from state s_i in *unif*($\mathcal{C}$). For $u_1 \in U_1 \setminus \{ s_1 \}$, we have:

$$\begin{aligned}
\overline{K_1} \cdot \sum_{u_2 \in U_2} \Delta(u_1, u_2) &= \frac{E(s_1)}{E} \cdot K_1 \cdot \sum_{u_2 \in U_2} \Delta(u_1, u_2) \\
&= \frac{E(s_1)}{E} \cdot \delta_1(u_1) \cdot \mathbf{P}(s_1, u_1) \\
&= \delta_1(u_1) \cdot \frac{\mathbf{R}(s_1, u_1)}{E} \;=\; \overline{\delta_1}(u_1) \cdot \overline{\mathbf{P}}(s_1, u_1).
\end{aligned}$$

For $s_1 \in U_1$ it follows by easy verification that:

$$\overline{K_1} \cdot \sum_{u_2 \in U_2} \Delta(s_1, u_2) = \delta_1(s_1) \cdot \frac{\mathbf{R}(s_1, s_1)}{E} = \overline{\delta_1}(s_1) \cdot \overline{\mathbf{P}}(s_1, s_1)$$

In the same way, condition 2.(ii) can be proven for state s_2.

Rate condition. We have:

$$\begin{aligned}
\overline{K_1} \cdot E &= \sum_{u_1 \in U_1} \overline{\delta_1}(u_1) \cdot \overline{\mathbf{R}}(s_1, u_1) \\
&= \sum_{\substack{u_1 \in U_1 \\ u_1 \neq s_1}} \delta_1(u_1) \cdot \mathbf{R}(s_1, u_1) \;+\; \delta_1(s_1) \cdot \frac{\mathbf{R}(s_1, s_1)}{\overline{\mathbf{R}}(s_1, s_1)} \cdot \overline{\mathbf{R}}(s_1, s_1) \\
&= \sum_{\substack{u_1 \in U_1 \\ u_1 \neq s_1}} \delta_1(u_1) \cdot \mathbf{R}(s_1, u_1) \;+\; \delta_1(s_1) \cdot \mathbf{R}(s_1, s_1) \\
&= \sum_{u_1 \in U_1} \delta_1(u_1) \cdot \mathbf{R}(s_1, u_1) \;=\; K_1 \cdot E(s_1)
\end{aligned}$$

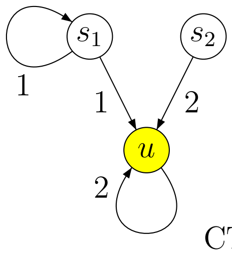
By a similar argument it follows that $\overline{K_2} \cdot E = K_2 \cdot E(s_2)$. Since $K_1 \cdot E(s_1) \leqslant K_2 \cdot E(s_2)$ we thus have $\overline{K_1} \cdot E \leqslant \overline{K_2} \cdot E$.

($\Leftarrow$) The converse direction can be shown in a similar way.

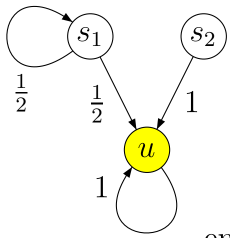
■

Note that the proof of the last part of the previous proposition (as well as the proof for the transitivity of $\precapprox_c$) relies on the fact that sets U_i and V_i may overlap. A few further remarks are in order.

Although $\precsim_c$ and $\precsim_d$ coincide for uniformized CTMCs (as $\precsim_c$ agrees with $\sim_c$, $\sim_c$ agrees with $\sim_d$, and $\sim_d$ agrees with $\precsim_d$), this does not hold for $\precapprox_d$ and $\precapprox_c$. For example, in:



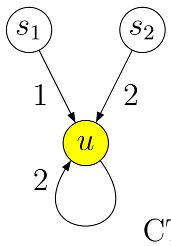
CTMC



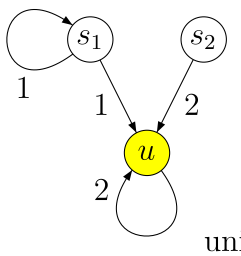
embedded DTMC

$s_2 \precapprox_d s_1$ in the embedded DTMC (on the right), but $s_2 \not\precapprox_c s_1$ in the uniformized CTMC (on the left), as the rate condition in Def. 39 is violated: $K_2 \cdot E(s_2) = 2 \not\leqslant 1 = K_1 \cdot E(s_1)$.

Secondly, note that the analogue of Prop. 43.3 (i.e., $\precapprox_c$ in $\mathcal{C}$ and $\precapprox_c$ in *unif*$(\mathcal{C})$ coincide) does not hold for the strong simulation preorder $\precsim_c$. This can be seen by considering the CTMC $\mathcal{C}$ and its uniformized CTMC *unif*$(\mathcal{C})$ in the picture below. Here, we have $s_1 \precsim_c s_2$ in $\mathcal{C}$, but $s_1 \not\precsim_c s_2$ in *unif*$(\mathcal{C})$.

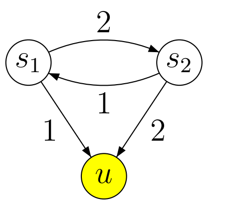


CTMC



uniformized CTMC

Finally, we note that although for uniformized CTMCs, $\sim_c$ and $\approx_c$ agree, a similar result for the simulation preorders does not hold. An example CTMC for which $s_1 \precapprox_c s_2$ but $s_1 \not\precsim_c s_2$ is:



The fact that $s_1 \not\precsim_c s_2$ follows from the weight function condition in Def. 22, e.g., the distribution to move to the u- and v-states are different $\mathbf{P}(s_1, \{ u \}) = \frac{1}{3} \neq \frac{2}{3} = \mathbf{P}(s_2, \{ u \})$.

To see that $s_1 \precsim_c s_2$, consider the reflexive closure R of $\{ (s_1, s_2) \}$ and the partitioning $V_1 = \{ s_2 \}$, $V_2 = \{ s_1 \}$ and $U_1 = U_2 = \{ u \}$ for which the conditions of a weak simulation are fulfilled.

3.5. Weak simulation equivalence

For the strong relations on FPSs or CTMCs, simulation equivalence agrees with bisimulation equivalence. For the equivalences $\precsim_d \cap \precsim_d^{-1}$ and $\precsim_c \cap \precsim_c^{-1}$, also denoted by $\cong_d$ and $\cong_c$, respectively, a similar relationship with $\approx_d$ and $\approx_c$ can be established. Recall that due to the reachability (and rate) condition, the weak simulation preorder on FPSs (or DTMCs) and CTMCs is non-symmetric. In particular, $\precsim_*$ is strictly coarser than weak simulation equivalence $\cong_*$ and $\approx_*$ where $* \in \{c, d\}$. The latter, however, coincide by the following theorem. We first consider the following proposition:

Proposition 44. For CTMC $\mathcal{C}$ with $s_1 \precsim_c s_2$ and $s_2 \precsim_c s_1$:

$$(s_1, s_2 \notin U \subseteq S \textit{ and } (U = U\!\uparrow \textit{ or } U = U\!\downarrow)\ ) \textit{ implies } \mathbf{R}(s_1, U) = \mathbf{R}(s_2, U).$$

Here, $\downarrow = \downarrow_{\precsim_c}$ and $\uparrow = \uparrow_{\precsim_c}$, i.e., U is downward- or upward-closed with respect to $\precsim_c$.

Proof: Assume $U = U\!\uparrow$. (The proof for $U = U\!\downarrow$ goes along the same lines.) In the sequel, we write $\cong_c$ to denote the weak simulation equivalence, i.e., $\cong_c = \precsim_c \cap \precsim_c^{-1}$. Let s_1, $s_2 \in S \setminus U$ with $s_1 \cong_c s_2$. Note $U \cap [s_1]_{\cong_c} = \varnothing$. We show that $\mathbf{R}(s_1, U) \leqslant \mathbf{R}(s_2, U)$. By symmetry, $\mathbf{R}(s_2, U) \leqslant \mathbf{R}(s_1, U)$, and thus $\mathbf{R}(s_1, U) = \mathbf{R}(s_2, U)$. In case $\mathsf{Post}(s_1) \subseteq s_2\!\downarrow$ (i.e., $K_1 = 0$) we have $\mathsf{Post}(s_1) \cap U = \varnothing$; otherwise $s_2 \in U\!\uparrow = U$ which contradicts the assumption that $s_2 \notin U$. Hence, $\mathbf{R}(s_1, U) = 0 \leqslant \mathbf{R}(s_2, U)$. In other cases, there exist δ_i, U_i, V_i, K_i, Δ, as in Def. 39 where $K_1 > 0$. Then, also $K_2 > 0$, since $K_1 \cdot E(s_1) \leqslant K_2 \cdot E(s_2)$. Moreover, we have:

$$v \in V_1 \implies v \, R \, s_2 \implies v \precsim_c s_2 \implies v \notin U$$

because $v \in V_1 \cap U$ would imply that $s_2 \in U = U\!\uparrow$. Hence, $\mathsf{Post}(s_1) \cap U \subseteq U_1$ and $\delta_1(u) = 1$ for all $u \in \mathsf{Post}(s_1) \cap U$. We now derive:

$$\begin{aligned}
\mathbf{R}(s_1, U) &= E(s_1) \cdot \textstyle\sum_{u \in U} \mathbf{P}(s_1, u) \\
&= E(s_1) \cdot \textstyle\sum_{u \in U} K_1 \cdot \sum_{u_2 \in S} \underbrace{\Delta(u, u_2)}_{= 0, \text{ if } u_2 \notin U} \\
&= E(s_1) \cdot K_1 \cdot \textstyle\sum_{u \in U} \sum_{u_2 \in U} \Delta(u, u_2) \\
&= E(s_1) \cdot K_1 \cdot \textstyle\sum_{u_2 \in U} \sum_{u \in U} \Delta(u, u_2) \\
&\leqslant E(s_1) \cdot K_1 \cdot \textstyle\sum_{u_2 \in U} \sum_{u \in S} \Delta(u, u_2) \\
&= E(s_1) \cdot K_1 \cdot \textstyle\sum_{u_2 \in U} \underbrace{\delta_2(u_2)}_{\leqslant 1} \cdot \frac{\mathbf{P}(s_2, u_2)}{K_2} \\
&\leqslant \underbrace{\frac{E(s_1) \cdot K_1}{K_2}}_{\leqslant E(s_2)} \cdot \underbrace{\sum_{u_2 \in U} \mathbf{P}(s_2, u_2)}_{= \mathbf{P}(s_2, U)} \\
&\leqslant E(s_2) \cdot \mathbf{P}(s_2, U) \ = \ \mathbf{R}(s_2, U)
\end{aligned}$$

■

Similarly, for FPS $\mathcal{D}$ we obtain: if $s_1 \cong_d s_2$ and $U \subseteq S$ is downward- or upward-closed wrt. $\precapprox_d$, then

$$\frac{\mathbf{P}(s_1, U)}{1 - \mathbf{P}(s_1, B)} = \frac{\mathbf{P}(s_2, U)}{1 - \mathbf{P}(s_2, B)}$$

where $B = [s_1]_{\cong_d} = [s_2]_{\cong_d}$, and where we assume that $\mathbf{P}(s_i, B) < 1$ for i=1, 2. In addition, the reachability condition for $\precapprox_d$ (cf. Def. 34) ensures that for any weak simulation equivalence class B either all states in B can reach a state outside B or none of them can.

Theorem 45.

1. *For any FPS, weak simulation equivalence* $\precapprox_d \cap \precapprox_d^{-1}$ *coincides with* $\approx_d$.
2. *For any CTMC, weak simulation equivalence* $\precapprox_c \cap \precapprox_c^{-1}$ *coincides with* $\approx_c$.

Proof: We prove the latter statement; the proof of the first statement is conducted similarly. Let $\mathcal{C} = (S, \mathbf{R}, L)$ be a CTMC. As before, we write $\cong_c$ to denote the weak simulation equivalence, i.e., $\cong_c = \precapprox_c \cap \precapprox_c^{-1}$. By Prop. 43.2, $\cong_c$ is coarser than $\approx_c$. It remains to prove the reverse, i.e., $\cong_c$ is a weak bisimulation on $\mathcal{C}$. Let $s_1 \cong_c s_2$. Clearly, $L(s_1) = L(s_2)$. We show:

$$\mathbf{R}(s_1, C) = \mathbf{R}(s_2, C) \text{ for all } C \in S/\cong_c \text{ with } C \neq [s_1]_{\cong_c} = [s_2]_{\cong_c}.$$

Let $B, C \in S/\cong_c$, $B \neq C$ and $B = [s_1]_{\cong_c} = [s_2]_{\cong_c}$. Distinguish the following cases:

- $C \not\precapprox_c B$, i.e., no state in C is weakly simulated by some state in B. Then, $s_1, s_2 \notin C\!\uparrow$ and $s_1, s_2 \notin C\!\uparrow \setminus C$. We derive using Prop. 44:

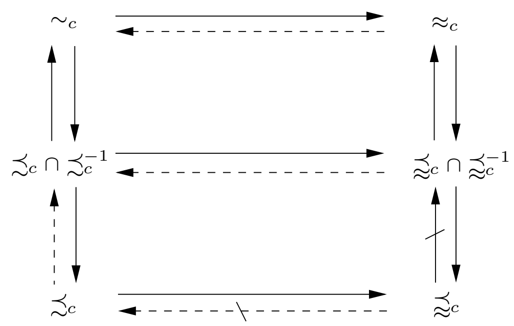
  $$\mathbf{R}(s_1, C\!\uparrow) - \mathbf{R}(s_1, C) = \mathbf{R}(s_1, C\!\uparrow \setminus C) = \mathbf{R}(s_2, C\!\uparrow \setminus C) = \mathbf{R}(s_2, C\!\uparrow) - \mathbf{R}(s_2, C)$$

  As, by Prop. 44, $\mathbf{R}(s_1, C\!\uparrow) = \mathbf{R}(s_2, C\!\uparrow)$, it follows $\mathbf{R}(s_1, C) = \mathbf{R}(s_2, C)$.

- $C \precapprox_c B$, i.e., there do exist states in C that are weakly simulated by a state in B. Then $C\!\downarrow \cap B = \varnothing$, as $C \cap B = \varnothing$. The proof of $\mathbf{R}(s_1, C) = \mathbf{R}(s_2, C)$ is conducted as in the previous case using $C\!\downarrow$ rather than $C\!\uparrow$.

■

Summarizing the results for the (bi)simulation equivalences and simulation preorders yields the following spectrum for the continuous-time setting. For the discrete-time setting, a similar figure is obtained.



Recall that an arrow from relation R to R' means that R is finer than R' whereas a "negated" arrow denotes that R' is not finer than R. The dashed arrows refer to uniformized CTMCs; note that for this special class of CTMCs all relations except $\precapprox_c$ coincide.

4. Logical characterizations

In the previous section, strong and weak (bi)simulation relations have been introduced for the discrete- and continuous-time setting, and their relationship has been studied. The focus of this section is on establishing logical characterizations of these relations. This will be done using the logics PCTL (Probabilistic CTL [35]) and CSL (Continuous Stochastic Logic [5,8]) for the discrete and continuous case, respectively. PCTL and CSL are both extensions of the branching-time temporal logic CTL (Computation Tree Logic). As these logics are widely used for model checking of probabilistic systems, establishing logical characterizations of the (bi)simulation relations is of particular interest. For instance, for the bisimulation relations it will be shown that they coincide with logical equivalence on either PCTL or CSL (or a fragment thereof). On the one hand, these results can be exploited for model checking by reducing (according to the appropriate bisimulation relation) the probabilistic models under consideration prior to carrying out the verification. This may speed up the verification as (mostly) a smaller model needs to be checked. On the other hand, this result allows for demonstrating that two probabilistic models are not bisimilar by providing a single PCTL- or CSL-formula that holds for one of the models but not for the other. For simulation relations, weak preservation results will be established that formalize the intuition that when s' simulates s, then s' is more "safe" than s. The notion of more "safe" is defined by a preorder on a (safe) fragment of the logic at hand. We start by defining some preliminary concepts that are needed to establish these results.

4.1. Computation paths

Paths in FPSs

A path corresponds to an execution or run of the system. Intuitively, a path in an FPS is a maximal sequence of states obtained by traversing the edge relation of the underlying graph of the FPS. Maximality means that the path is either infinite or finite and ends in an absorbing or sub-stochastic state. To distinguish the prefix $s_0, s_1, \ldots, s_n$ of a path that continues in sub-stochastic state s_n from the path that stays forever in state s_n, any finite path is required to end with the symbol $\perp$.

Definition 46. Let $\mathcal{D} = (S, \mathbf{P}, L)$ be a FPS.

- An *infinite path* σ in $\mathcal{D}$ is an infinite sequence $s_0, s_1, s_2, \ldots$ of states such that $\mathbf{P}(s_i, s_{i+1}) > 0$ for all $i \geqslant 0$.
- A *finite path* σ in $\mathcal{D}$ is a sequence $s_0, s_1, \ldots, s_n, \perp$ such that $\mathbf{P}(s_i, s_{i+1}) > 0$ for $0 \leqslant i < n$, and $\mathbf{P}(s_n, \perp) > 0$.
- A *path fragment* is a (possibly non-maximal) portion of a path in $\mathcal{D}$, i.e., a sequence $s_0, s_1, \ldots, s_n$ such that $s_n \in S_\perp$ and $\mathbf{P}(s_i, s_{i+1}) > 0$ for $0 \leqslant i < n$.

Path(s) denotes the set of all (finite and infinite) paths that start in state s. ■

Note that any path in a DTMC (i.e., FPS with only stochastic states) is infinite. Let $|\sigma|$ denote the length of a path or path fragment σ, i.e., $|s_0, s_1, \ldots, s_n| = |s_0, s_1, \ldots, s_n, \perp| = n$ and $|\sigma| = \infty$ for infinite σ. For $i \leqslant |\sigma|$, $\sigma[i] = s_i$ denotes the $(i{+}1)$-st state in σ.

Any FPS $\mathcal{D}$ enriched with a start state s induces a probability space. The underlying sigma-algebra is generated from the basic cylinders induced by the finite path fragments starting in s. The probability measure $\Pr_s^{\mathcal{D}}$ (briefly Pr) induced by $(\mathcal{D}, s)$ is the unique measure on this sigma-algebra where

$$\Pr\underbrace{\{\sigma \in \mathit{Path}(s) \mid s = s_0, s_1, \ldots, s_n \text{ is a prefix of } \sigma\}}_{\substack{\text{basic cylinder of the} \\ \text{path fragment } s, s_1, \ldots, s_n}} = \prod_{0 \leqslant i < n} \mathbf{P}(s_i, s_{i+1}).$$

Observe that if $s_n = \bot$, the basic cylinder induced by $\sigma = s, s_1, \ldots, s_{n-1}, s_n$ just consists of σ.

Paths in CTMCs

A path in a CTMC is similar to a path in an FPS except that for each visited state its residence time is recorded. Formally, paths in a CTMC are maximal alternating sequences $s_0, t_0, s_1, t_1, s_2, \ldots$ that are either infinite or end in an absorbing state.

Definition 47. Let $\mathcal{C} = (S, \mathbf{R}, L)$ be a CTMC.

- An *infinite path* σ in $\mathcal{C}$ is an infinite sequence $s_0 \xrightarrow{t_0} s_1 \xrightarrow{t_1} s_2 \xrightarrow{t_2} \ldots$ with $s_i \in S$ and $t_i \in \mathbb{R}_{>0}$ such that $\mathbf{R}(s_i, s_{i+1}) > 0$ for all $i \geqslant 0$.
- A *finite path* σ in $\mathcal{C}$ is a sequence $s_0 \xrightarrow{t_0} s_1 \xrightarrow{t_1} \ldots s_{n-1} \xrightarrow{t_{n-1}} s_n$ such that s_n is absorbing, and $\mathbf{R}(s_i, s_{i+1}) > 0$ for $0 \leqslant i < n$.

■

The notations $\mathit{Path}(s)$, $\sigma[i]$ and $|\sigma|$ are as for paths in FPSs. For infinite path σ and $i \geqslant 0$, let $\delta(\sigma, i) = t_i$, the time spent in s_i. For $t \in \mathbb{R}_{\geqslant 0}$ and i the smallest index with $t \leqslant \sum_{j=0}^{i} t_j$ let $\sigma@t = \sigma[i]$, the state in σ occupied at time t. For finite σ that ends in s_n, $\sigma[i]$ and $\delta(\sigma, i)$ are only defined for $i \leqslant n$; they are defined for $i < n$ in the above way, and $\delta(\sigma, n) = \infty$. For $t > \sum_{j=0}^{n-1} t_j$ let $\sigma@t = s_n$; otherwise, $\sigma@t$ is as above.

Similar to the discrete-time case, basic cylinders, a sigma-algebra, and a unique probability measure over paths can be defined; for details, see [8]. In the sequel, $\Pr_s^{\mathcal{C}}$, or simply Pr, denotes the unique probability measure on sets of paths in CTMC $\mathcal{C}$ (that start in a state s).

4.2. Probabilistic Computation Tree Logic

Probabilistic CTL (PCTL) [35] is a probabilistic extension of CTL in which state-formulae are interpreted over states of an FPS and path-formulae are interpreted over paths in an FPS. In PCTL, the universal and existential path quantifiers of (fair) CTL are replaced by a single probability operator, denoted $\mathcal{P}$, which allows to refer to the probability of the occurrence of particular paths. For example, for path-formula φ, the state-formula $\mathcal{P}_{>p}(\varphi)$ holds in state s if and only if the probability of all paths satisfying φ that start in s exceeds probability p. A $\mathcal{P}$-formula thus has three parameters: a path-formula characterizing the paths of interest, a probability, and a comparison operator.

Path-formulae are constructed using the standard next- and until-operator [4]. To simplify the definition of a safe fragment of PCTL later on, we consider here formulae in positive normal form, which means that negation only occurs on the level of literals. To retain the power of PCTL with "full" negation, for the temporal operators X (next step) and $\mathcal{U}$ (until), we insert the weak variants $\widetilde{X}$ (weak next step) and $\widetilde{\mathcal{U}}$ (weak until).

Syntax

Let probability $p \in [0,1]$ and $\trianglelefteq$ a binary comparison operator, i.e., $\trianglelefteq \in \{<, \leqslant, \geqslant, >\}$. Recall that AP denotes a fixed, finite set of atomic propositions ranged over by $a, b, c, \ldots$. The syntax of PCTL state-formulae (in positive normal form) is defined as follows:

$$\Phi ::= \mathrm{tt} \;\Big|\; a \;\Big|\; \neg\, a \;\Big|\; \Phi \wedge \Phi \;\Big|\; \Phi \vee \Phi \;\Big|\; \mathcal{P}_{\trianglelefteq p}(\varphi)$$

where φ is a path-formula defined according to the following grammar:

$$\varphi ::= X\Phi \;\Big|\; \widetilde{X}\,\Phi \;\Big|\; \Phi\,\mathcal{U}\,\Phi \;\Big|\; \Phi\,\widetilde{\mathcal{U}}\,\Phi\,.$$

The propositional fragment of PCTL has the usual interpretation. $\mathcal{P}_{\trianglelefteq p}(\varphi)$ asserts that the probability measure of the paths satisfying φ meets the bound given by $\trianglelefteq p$. The intuitive meaning of $X\Phi$ is that Φ will hold in the next state. $\widetilde{X}$ is its weak counterpart, and does not the existence of a next step. For instance, $\mathcal{P}_{\geqslant 0.9}(\widetilde{X}\,a)$ states that with at least probability 0.9, either no next state is reached or a next state not satisfying a is reached. Stated differently, with probability less than 0.1 the next state does satisfy a. Thus, $\mathcal{P}_{\leqslant 0.9}(\widetilde{X}\,a)$ is equivalent to $\mathcal{P}_{>0.1}(Xa)$. The path-formula $\Phi\,\mathcal{U}\,\Psi$ asserts that Ψ eventually holds and that at all preceding states Φ holds (strong until). For instance, the formula $\mathcal{P}_{\geqslant 0.91}(\mathit{green}\,\mathcal{U}\,\mathit{red})$ states that the probability to eventually reach a red state via a path of green states is at least 0.91. $\widetilde{\mathcal{U}}$ is its weak counterpart and does not require Ψ to eventually become true. For instance, $\mathcal{P}_{\geqslant 0.91}(\mathit{green}\,\widetilde{\mathcal{U}}\,\mathit{red})$ asserts that the probability of either staying green forever, or reaching a red state via a green path, is at least 0.91. Stated differently, with probability less than 0.09, a state is reached that is neither red nor green via a path that does not contain a red state.

As for CTL, temporal operators like $\Diamond$ (eventually) and $\Box$ (always) can be derived, e.g.

$$\mathcal{P}_{\trianglelefteq p}(\Diamond\,\Phi) = \mathcal{P}_{\trianglelefteq p}(\mathrm{tt}\,\mathcal{U}\,\Phi) \;\text{ and }\; \mathcal{P}_{\trianglelefteq p}(\Box\,\Phi) = \mathcal{P}_{\trianglelefteq p}(\Phi\,\widetilde{\mathcal{U}}\,\mathrm{ff})$$

where ff equals $a \wedge \neg a$. For instance, if *error* is an atomic proposition that characterizes all states where a system error has occurred then $\mathcal{P}_{\leqslant 0.001}(\Diamond\,\mathit{error})$ asserts that the probability for a system error to occur eventually is at most 10^{-3}.

Semantics

Let FPS $\mathcal{D} = (S, \mathbf{P}, L)$. The semantics of PCTL is defined by a satisfaction relation, denoted $\models$, which is characterized as the least relation over the states in S (paths in

[4]In this paper, the bounded until-operator [35] is omitted. Although the logical characterization results for the strong (bi)simulation relations also hold when this operator is incorporated, for the weak relations this is not the case as these relations allow for stuttering.

$\mathcal{D}$, respectively) and the state formulae (path formulae). The semantics of the propositional fragment is identical to that for CTL. The meaning of the probabilistic operator is formalized as follows [35]. The semantics of PCTL state-formulae thus is defined for path-formula φ as:

$$
\begin{array}{llllll}
s \models \mathrm{tt} & & & s \models \Phi \wedge \Psi & \text{iff} & s \models \Phi \text{ and } s \models \Psi \\
s \models a & \text{iff} & a \in L(s) & s \models \Phi \vee \Psi & \text{iff} & s \models \Phi \text{ or } s \models \Psi \\
s \models \neg a & \text{iff} & a \notin L(s) & s \models \mathcal{P}_{\unlhd p}(\varphi) & \text{iff} & \mathrm{Pr}(s, \varphi) \unlhd p\,.
\end{array}
$$

Here, $\mathrm{Pr}(s, \varphi) = \mathrm{Pr}\{\, \sigma \in \mathit{Path}(s) \mid \sigma \models \varphi \,\}$ denotes the probability of the set of paths satisfying φ that start in s. The meaning of the path-operators is as for CTL. Let σ be a path in $\mathcal{D}$. The semantics of the PCTL path-formulae is defined as:

$$
\begin{array}{lll}
\sigma \models X\Phi & \text{iff} & |\sigma| \geqslant 1 \text{ and } \sigma[1] \models \Phi \\
\sigma \models \widetilde{X}\,\Phi & \text{iff} & \text{either } |\sigma| < 1 \text{ or } \sigma[1] \not\models \Phi \\
\sigma \models \Phi\,\mathcal{U}\,\Psi & \text{iff} & \sigma[i] \models \Phi,\ i = 0, 1, \ldots, n{-}1, \text{ and } \sigma[n] \models \Psi \text{ for some } n \leqslant |\sigma| \\
\sigma \models \Phi\,\widetilde{\mathcal{U}}\,\Psi & \text{iff} & \text{either } \sigma \models \Phi\,\mathcal{U}\,\Psi \text{ or } \sigma[i] \models \Phi \text{ for all } i \leqslant |\sigma|
\end{array}
$$

Recall that in FPSs, paths are either infinite or of the form $\sigma = s_0, s_1, \ldots, s_n, \bot$. In the latter case, $|\sigma| = n$ and $\sigma \models \Phi\,\widetilde{\mathcal{U}}\,\Psi$ iff either there exists $j \leqslant n$ such that $s_j \models \Psi$ and $s_i \models \Phi$ for $0 \leqslant i < j$, or $s_i \models \Phi$ for $0 \leqslant i \leqslant n$.

The next (until)-operator and the weak next (until)-operator are closely related. This follows from the following equations where for the sake of comparison we allow arbitrary state-formula to be negated. For any state s and all PCTL-formulae Φ and Ψ we have:

$$\mathrm{Pr}(s, X\Phi) = 1 - \mathrm{Pr}(s, \widetilde{X}\,\neg\Phi) \tag{2}$$

$$\mathrm{Pr}(s, \widetilde{X}\,\Phi) = 1 - \mathrm{Pr}(s, X\neg\Phi) \tag{3}$$

$$\mathrm{Pr}(s, \Phi\,\mathcal{U}\,\Psi) = 1 - \mathrm{Pr}\Big(s, (\neg\Psi)\,\widetilde{\mathcal{U}}\,\neg(\Phi \vee \Psi)\Big) \tag{4}$$

$$\mathrm{Pr}(s, \Phi\,\widetilde{\mathcal{U}}\,\Psi) = 1 - \mathrm{Pr}(s, (\neg\Psi)\,\mathcal{U}\,\neg(\Phi \vee \Psi)) \tag{5}$$

Hence, the following pairs of formulae are equivalent:

$$
\begin{array}{lll}
\mathcal{P}_{\geqslant p}(X\Phi) & \equiv & \mathcal{P}_{\leqslant 1-p}(\widetilde{X}\,\neg\Phi) \\
\mathcal{P}_{\geqslant p}(\widetilde{X}\,\Phi) & \equiv & \mathcal{P}_{\leqslant 1-p}(X\neg\Phi) \\
\mathcal{P}_{\geqslant p}(\Phi\,\widetilde{\mathcal{U}}\,\Psi) & \equiv & \mathcal{P}_{\leqslant 1-p}((\neg\Psi)\,\mathcal{U}\,\neg(\Phi \vee \Psi)) \\
\mathcal{P}_{\geqslant p}(\neg\,\Phi\,\mathcal{U}\,\neg\,\Psi) & \equiv & \mathcal{P}_{\leqslant 1-p}(\Psi\,\widetilde{\mathcal{U}}\,(\Phi \wedge \Psi)).
\end{array}
$$

In particular, these equivalences show how any PCTL-formula (with "full" negation) can be transformed into positive normal form.

4.3. Continuous Stochastic Logic

Continuous Stochastic Logic (CSL) [8] is a variant of the (identically named) logic by Aziz *et al.* [5] and extends PCTL by path operators that reflect the real-time nature of CTMCs: a time-bounded next- and until-operator. To be able to reason about the equilibrium behaviour of a CTMC, a steady-state operator $\mathcal{S}$ is introduced [5]. For example,

[5]In a similar way, PCTL could be extended with a long-run operator that allows the specification of properties about the long-run behaviour of FPSs.

for state-formula Φ, $\mathcal{S}_{>p}(\Phi)$ holds in state s if and only if the probability to be in the long run in some Φ-state when started in s exceeds p. We focus here on a fragment of CSL where the time bounds of (weak) until are of the form "$\leqslant t$"; other time bounds can be handled by mappings on this case [8].

Syntax

Let p and $\trianglelefteq$ as before. The syntax of CSL state-formulae (in positive normal form) is defined as follows.

$$\Phi ::= \mathrm{tt} \;\Big|\; a \;\Big|\; \neg\, a \;\Big|\; \Phi \wedge \Phi \;\Big|\; \Phi \vee \Phi \;\Big|\; \mathcal{S}_{\trianglelefteq p}(\Phi) \;\Big|\; \mathcal{P}_{\trianglelefteq p}(\varphi)$$

where φ is a path-formula defined, for t a non-negative real number or ∞, according to the following grammar:

$$\varphi ::= X^{\leqslant t}\Phi \;\Big|\; \widetilde{X}^{\leqslant t}\Phi \;\Big|\; \Phi\,\mathcal{U}^{\leqslant t}\,\Phi \;\Big|\; \Phi\,\widetilde{\mathcal{U}}^{\leqslant t}\,\Phi\,.$$

Compared to PCTL, the next- and until-operators are equipped with a time bound. The intuitive meaning of $X^{\leqslant t}\Phi$ is that Φ holds in the next state and is reached within t time units. Similarly, the path-formula $\Phi\,\mathcal{U}^{\leqslant t}\,\Psi$ asserts that Ψ is satisfied at some time instant before or equal to t and that at all preceding time instants Φ holds. The connection between the until-operator and the weak until-operator is as in PCTL. As for PCTL, temporal operators like $\Diamond^{\leqslant t}$ (eventually within time t) and $\Box^{\leqslant t}$ can be derived.

Semantics

CSL state-formulas are interpreted over the states of a CTMC. Let $\mathcal{C} = (S, \mathbf{R}, L)$ with labels in AP, and $Sat(\Phi) = \{\, s \in S \mid s \models \Phi \,\}$ the set of states satisfying the state-formula Φ. The semantics of CSL state-formulae is defined for path-formula φ as:

$$\begin{array}{lll@{\qquad}lll}
s \models \mathrm{tt} & & & s \models \Phi \wedge \Psi & \text{iff} & s \models \Phi \text{ and } s \models \Psi \\
s \models a & \text{iff} & a \in L(s) & s \models \Phi \vee \Psi & \text{iff} & s \models \Phi \text{ or } s \models \Psi \\
s \models \neg a & \text{iff} & a \notin L(s) & s \models \mathcal{S}_{\trianglelefteq p}(\Phi) & \text{iff} & \pi(s, Sat(\Phi)) \trianglelefteq p \\
 & & & s \models \mathcal{P}_{\trianglelefteq p}(\varphi) & \text{iff} & \Pr(s, \varphi) \trianglelefteq p\,.
\end{array}$$

Here, $\Pr(s, \varphi)$ is as defined for PCTL (referring to paths in $\mathcal{C}$, of course), and $\pi(s, S')$ for $S' \subseteq S$ denotes the steady-state probability [34,49,61] for S' when starting in state s, i.e.,

$$\pi(s, S') \;=\; \lim_{t \to \infty} \Pr\{\, \sigma \in Path(s) \mid \sigma@t \in S' \,\}.$$

For path σ in $\mathcal{C}$, the satisfaction relation for CSL path-formulae is defined as:

$$\begin{array}{ll}
\sigma \models X^{\leqslant t}\Phi & \text{iff } \sigma[1] \text{ is defined and } \sigma[1] \models \Phi \text{ and } \delta(\sigma, 0) \leqslant t \\
\sigma \models \widetilde{X}^{\leqslant t}\Phi & \text{iff either } |\sigma| < 1 \text{ or } \sigma[1] \not\models \Phi \text{ or } \delta(\sigma, 0) > t \\
\sigma \models \Phi\,\mathcal{U}^{\leqslant t}\,\Psi & \text{iff } \sigma@x \models \Psi \text{ for some } x \leqslant t \text{ and } \sigma@y \models \Phi \text{ for all } y < x \\
\sigma \models \Phi\,\widetilde{\mathcal{U}}^{\leqslant t}\,\Psi & \text{iff either } \sigma \models \Phi\,\mathcal{U}^{\leqslant t}\,\Psi \text{ or } \sigma@x \models \Phi \text{ for all } x \leqslant t
\end{array}$$

Note that $\Phi\,\mathcal{U}\,\Psi$ can be interpreted as an abbreviation of $\Phi\,\mathcal{U}^{\leqslant \infty}\,\Psi$. The relationship between the next (until)-operator and their weak counterparts is the same as for PCTL.

4.4. Logical characterization of weak bisimulation

In both the discrete and the continuous setting, strong bisimulation ($\sim_d$ and $\sim_c$) coincide with logical equivalence (in PCTL and CSL, respectively). The latter are denoted $\equiv_{\text{PCTL}}$ and $\equiv_{\text{CSL}}$, respectively. That is, $s_1 \equiv_{\text{PCTL}} s_2$ iff s_1 and s_2 satisfy exactly the same PCTL formulae. Similarly, $s_1 \equiv_{\text{CSL}} s_2$ iff s_1 and s_2 satisfy exactly the same CSL formulae.

Theorem 48. [4] *For any FPS: $\sim_d$ coincides with $\equiv_{PCTL}$.*

Note that [4] shows that $\sim_d$ coincides with PCTL*-equivalence where PCTL* is a logic that subsumes PCTL and allows for, for instance, the conjunction of path formulae and arbitrary combination of modalities. In order to establish a logical characterization of $\sim_d$, it turns out that a fragment of PCTL without the until-operators is sufficient. Desharnais *et al.* [25] have shown that even conjunction and probabilistic next suffice for that purpose.

Theorem 49. [8,28] *For any CTMC: $\sim_c$ coincides with $\equiv_{CSL}$.*

The paper [28] shows that $\sim_c$ and $\equiv_{\text{CSL}}$ not only coincide for CTMCs with a countable state space but also for continuous-state processes.

In the rest of this section, we focus on establishing strong preservation results for weak bisimulation and the fragments of the logics PCTL and CSL without next (and weak next). The next-operators are omitted as they are not stutter-invariant, and thus it is impossible to establish a strong preservation result for weak (bi)simulation in the presence of these operators. Let $\text{PCTL}_{\setminus X}$ denote the fragment of PCTL without the next-step and the weak next-step operator; similarly, $\text{CSL}_{\setminus X}$ is defined. $\text{PCTL}_{\setminus X}$-equivalence, denoted $\equiv_{\text{PCTL}_{\setminus X}}$, and $\text{CSL}_{\setminus X}$-equivalence, denoted $\equiv_{\text{CSL}_{\setminus X}}$, are defined in the obvious way.

Theorem 50. *For any FPS: $\approx_d$ coincides with $\equiv_{PCTL_{\setminus X}}$.*

Proof: (Soundness). The fact that $\approx_d$ implies $\equiv_{\text{PCTL}_{\setminus X}}$ is proven by structural induction on the syntax of $\text{PCTL}_{\setminus X}$-formulae. Let $s \approx_d s'$. The base cases tt, a and $\neg a$ are straightforward: all states satisfy tt (and thus s and s'), and a ($\neg a$) holds iff $a \in L(s) = L(s')$ ($a \notin L(s) = L(s')$). For conjunction (and disjunction) the proof directly follows from the induction hypotheses on the conjuncts (disjuncts, respectively). It remains to consider the until operator. The proof for the weak-until operator can be conducted in a similar way as for until and is omitted. Let $\varphi = \Phi\,\mathcal{U}\,\Psi$. For $s \approx_d s'$ we aim to establish that $\Pr(s,\varphi) = \Pr(s',\varphi)$, and thus $s \models \mathcal{P}_{\trianglelefteq p}(\varphi)$ iff $s' \models \mathcal{P}_{\trianglelefteq p}(\varphi)$. By the induction hypothesis it follows that both $Sat(\Phi)$ and $Sat(\Psi)$ are a disjoint union of equivalence classes under $\approx_d$. Let $B = [s]_{\approx_d}$. Then, $B \cap Sat(\Phi) = \varnothing$ or $B \subseteq Sat(\Phi)$ (and similar for Ψ). Only the cases $B \subseteq Sat(\Phi)$ and $B \cap Sat(\Psi) = \varnothing$ are of interest; for all other cases, $\Pr(s,\varphi) = \Pr(s',\varphi) \in \{\,0, 1\,\}$ and the theorem directly follows. Let S' be the set of states that can reach a Ψ-state via a (non-empty) Φ-path, i.e., $S' = \{\, s \mid \Pr(s,\varphi) > 0 \,\} \setminus Sat(\Psi)$. As $Sat(\Phi)$ and $Sat(\Psi)$ are disjoint unions of equivalence classes under $\approx$, S' can be viewed as such a disjoint union too.

For $s \notin S'$, $\Pr(s,\varphi) \in \{\,0,1\,\}$. For $s \in S'$, the vector $(\Pr(s,\varphi))_{s \in S'}$ is the *unique* solution of the linear equation system:

$$x_s = \mathbf{P}(s, Sat(\Psi)) \;+\; \sum_{s' \in S'} \mathbf{P}(s,s') \cdot x_{s'} \tag{6}$$

The first summand denotes the probability to go from state s to a Ψ-state in one step, whereas the second summand denotes the probability to go from s to a Ψ-state via at least one Φ-state. For any $\approx_d$-equivalence class $B \subseteq S'$, select $s_B \in B$ such that $\mathbf{P}(s_B, B) < 1$, i.e., s_B is a state via which B can be directly left. Stated differently, $s_B \notin \mathsf{Silent}_{\approx_d}$. Such state is guaranteed to exist, since if $\mathbf{P}(s,B)$ would equal 1 for all states $s \in B$ then none of the B-states can reach a Ψ-state, contradicting $B \subseteq S'$. Now consider the unique solution $(x_B)_{B \in S/\approx_d, B \subseteq S'}$ of the linear equation system:

$$x_B \;=\; \mathbf{P}(s_B, \mathit{Sat}(\Psi)) \;+\; \sum_{\substack{C \in S/\approx_d \\ C \subseteq S'}} \mathbf{P}(s_B, C) \cdot x_C$$

We now show that $x_s = x_B$ for all states $s \in B$. For this, we prove that the vector $(y_s)_{s \in S'}$ is a solution to (6) where $y_s = x_B$ if $s \in B$ and B ranges over all $\approx_d$-equivalence classes $B \subseteq S'$.

We first consider the case $s \in B$ and $\mathbf{P}(s,B) = 1$ and show that equation (6) for state s holds for the values $y_{s'}$ rather than $x_{s'}$. As $\mathbf{P}(s,s') = 0$ for all states $s' \in S \setminus B$, the sum on the right-hand side of equation (6) with $y_{s'}$ rather than $x_{s'}$ reduces to:

$$\sum_{s' \in B} \mathbf{P}(s,s') \cdot \underbrace{y_{s'}}_{=x_B} \;=\; x_B \cdot \underbrace{\sum_{s' \in B} \mathbf{P}(s,s')}_{=\mathbf{P}(s,B)=1} \;=\; x_B \;=\; y_s$$

Next we consider equation (6) for the states $s \in B$ where $\mathbf{P}(s,B) < 1$. By definition of $\approx_d$, we have

$$\frac{\mathbf{P}(s,C)}{1 - \mathbf{P}(s,B)} \;=\; \frac{\mathbf{P}(s_B,C)}{1 - \mathbf{P}(s_B,B)}$$

for all states $s \in B$ and equivalence classes $C \in S/\approx_d$ with $C \neq B$. Hence:

$$\mathbf{P}(s,C) \;=\; \frac{1 - \mathbf{P}(s,B)}{1 - \mathbf{P}(s_B,B)} \cdot \mathbf{P}(s_B,C)$$

As $\mathit{Sat}(\Psi)$ is the union of equivalence classes under $\approx_d$, we obtain:

$$\mathbf{P}(s,\mathit{Sat}(\Psi)) \;=\; \frac{1 - \mathbf{P}(s,B)}{1 - \mathbf{P}(s_B,B)} \cdot \mathbf{P}(s_B,\mathit{Sat}(\Psi))$$

Thus, the sum on the right-hand side of equation (6) with $y_{s'} = x_C$ for $s' \in C$ rather than $x_{s'}$ can be rewritten as follows:

$$\begin{aligned}
& \mathbf{P}(s,\mathit{Sat}(\Psi)) + \sum_{s' \in S'} \mathbf{P}(s,s') \cdot y_{s'} \\
= \;& \mathbf{P}(s,\mathit{Sat}(\Psi)) + \sum_{\substack{C \in S/\approx_d \\ C \subseteq S'}} \mathbf{P}(s,C) \cdot x_C \\
= \;& \frac{1 - \mathbf{P}(s,B)}{1 - \mathbf{P}(s_B,B)} \cdot \mathbf{P}(s_B,\mathit{Sat}(\Psi)) \;+\; \sum_{\substack{C \in S/\approx_d \\ C \neq B, C \subseteq S'}} \frac{1 - \mathbf{P}(s,B)}{1 - \mathbf{P}(s_B,B)} \mathbf{P}(s_B,C) \cdot x_C \;+\; \mathbf{P}(s,B) \cdot x_B \\
= \;& \frac{1 - \mathbf{P}(s,B)}{1 - \mathbf{P}(s_B,B)} \underbrace{\left(\mathbf{P}(s_B,\mathit{Sat}(\Psi)) \;+\; \sum_{\substack{C \in S/\approx_d \\ C \neq B, C \subseteq S'}} \mathbf{P}(s_B,C) \cdot x_C \right)}_{=x_B - \mathbf{P}(s_B,B) \cdot x_B} + \mathbf{P}(s,B) \cdot x_B \\
= \;& \frac{1 - \mathbf{P}(s,B)}{1 - \mathbf{P}(s_B,B)} (x_B - \mathbf{P}(s_B,B) \cdot x_B) \;+\; \mathbf{P}(s,B) \cdot x_B \\
= \;& (1 - \mathbf{P}(s,B)) \cdot x_B \;+\; \mathbf{P}(s,B) \cdot x_B \;=\; x_B
\end{aligned}$$

Hence, $y_s = x_B = x_s = \Pr(s, \varphi)$ for all states $s \in B$ and $B \in S/\approx_d$. Consequently, $s \models \mathcal{P}_{\trianglelefteq p}(\varphi)$ iff $s' \models \mathcal{P}_{\trianglelefteq p}(\varphi)$ for any state $s' \in B = [s]$.

(Completeness). The fact that $\equiv_{\text{PCTL}_{\setminus X}}$ implies $\approx_d$ is proven by using so-called master formulae for the equivalence classes induced by $\equiv_{\text{PCTL}_{\setminus X}}$. These formulae are defined as follows. If the FPS is finite-state then the state-formula

$$\Phi_C = \bigwedge_{D \neq C} \Phi_{C,D}$$

uniquely characterizes all C-states where $\Phi_{C,D}$ is defined by

$$C \subseteq Sat(\Phi_{C,D}) \quad \text{and} \quad D \cap Sat(\Phi_{C,D}) = \varnothing$$

for different equivalence classes C and D under $\equiv_{\text{PCTL}_{\setminus X}}$. (For infinite-state FPSs, approximations of master-formulae can be used [24]; for simplicity we consider the finite-state case only). Assume S to be finite and that any equivalence class C under $\equiv_{\text{PCTL}_{\setminus X}}$ is represented by a $\text{PCTL}_{\setminus X}$-formula Φ_C. We now check the conditions of $\approx_d$ (cf. Def. 26). Let $s_1 \equiv_{\text{PCTL}_{\setminus X}} s_2$, and $B = [s_1] = [s_2]$ under $\equiv_{\text{PCTL}_{\setminus X}}$.

1. For set of atomic propositions $A \subseteq AP$ consider the propositional $\text{PCTL}_{\setminus X}$-formula:

   $$\Phi_A = \bigwedge_{a \in A} a \;\wedge\; \bigwedge_{b \notin A} \neg b$$

   $s_1 \equiv_{\text{PCTL}_{\setminus X}} s_2$ implies $s_1 \models \Phi_A$ iff $s_2 \models \Phi_A$, and, hence, by definition of Φ_A, $L(s_1) = L(s_2)$.

2. For $\text{PCTL}_{\setminus X}$ equivalence class C with $B \neq C$, let $\varphi = \Phi_B \,\mathcal{U}\, \Phi_C$. As $s_1 \equiv_{\text{PCTL}_{\setminus X}} s_2$, we have $\Pr(s_1, \varphi) = \Pr(s_2, \varphi)$. If $\mathbf{P}(s_i, B) < 1$ for i=1, 2, then:

   $$\Pr(s_i, \varphi) \;=\; \frac{\mathbf{P}(s_i, C)}{1 - \mathbf{P}(s_i, B)}.$$

   This is justified as follows. If $\Pr(s_i, \varphi) = 0$, then $\mathbf{P}(s_i, C) = 0$. Otherwise, by instantiating the equation system in (6) with $S' = B$, $\Phi_2 = \Phi_C$, and $\Phi_1 = \Phi_B$, it can be verified that the vector with the values $x_s = \frac{\mathbf{P}(s,C)}{1-\mathbf{P}(s,B)}$ (for $s \in B$) is a solution.

3. s_1 can reach a state outside B iff $s_1 \models \mathcal{P}_{>0}(\diamond \neg \Phi_B)$, which is equivalent – as $s_1 \equiv_{\text{PCTL}_{\setminus X}} s_2$ – to $s_2 \models \mathcal{P}_{>0}(\diamond \neg \Phi_B)$, or equivalently, to the statement that s_2 can reach a state outside B.

Hence, we conclude that $s_1 \approx_d s_2$. ■

The next objective is to establish a strong preservation result for $\approx_c$ and $\equiv_{\text{CSL}_{\setminus X}}$. To that end, we use the observation (cf. Prop. 52) that $\approx_c$ in CTMCs $\mathcal{C}$ and *unif*$(\mathcal{C})$ coincides. This allows for replacing $\mathcal{C}$ by its uniformized counterpart. Using the facts that $\approx_c$ and $\sim_c$ coincide for uniformized CTMCs, and that $\sim_c$ coincides with $\equiv_{\text{CSL}}$ gives the desired result.

Proposition 51. For CTMC $\mathcal{C}$, s in $\mathcal{C}$, and $\text{CSL}_{\setminus X}$-formula Φ:

$$s \models \Phi \quad \textit{iff} \quad s \models \Phi \textit{ in unif}(\mathcal{C}).$$

Proof: By induction on the syntax of Φ. For the propositional fragment the result is obvious. For the $\mathcal{S}$- and $\mathcal{P}$-operator, we exploit the fact that steady-state and transient distributions in $\mathcal{C}$ and $unif(\mathcal{C})$ are identical (cf. [57]), and that the semantics of $\mathcal{U}^{\leqslant t}$ and $\widetilde{\mathcal{U}}^{\leqslant t}$ agrees with transient distributions [8]. ■

Proposition 52. For any uniformized CTMC: $\equiv_{CSL}$ coincides with $\equiv_{CSL_{\setminus X}}$.

Proof: The direction "⇒" is obvious. We prove the other direction. Assume CTMC $\mathcal{C}$ is uniformized and let s_1, s_2 be states in $\mathcal{C}$. From Prop. 11.2 and the logical characterizations of $\sim_c$ and $\sim_d$ it follows:

$$s_1 \equiv_{\mathrm{CSL}} s_2 \quad \text{iff} \quad s_1 \sim_c s_2 \quad \text{iff} \quad s_1 \sim_d s_2 \quad \text{iff} \quad s_1 \equiv_{\mathrm{PCTL}} s_2.$$

By showing that $\equiv_{\mathrm{CSL}_{\setminus X}}$ implies $\equiv_{\mathrm{PCTL}}$ (for uniformized CTMC) we thus obtain the desired result. This is done by structural induction on the syntax of PCTL-formulae. Clearly, only the next step operator is of interest (the proof for weak next goes along similar lines and is omitted here). As in the proof of Theorem 50 we assume a finite state space and that any $\equiv_{\mathrm{CSL}_{\setminus X}}$-equivalence class C can be characterized by $\mathrm{CSL}_{\setminus X}$ formula Φ_C. Consider PCTL-path formula $\varphi = X\Phi$. By induction hypothesis, $Sat(\Phi)$ is a (countable) union of equivalence classes of $\equiv_{\mathrm{CSL}_{\setminus X}}$. In the following, we establish for $s_1 \equiv_{\mathrm{CSL}_{\setminus X}} s_2$:

$$\mathbf{P}(s_1, Sat(\Phi)) = \mathbf{P}(s_2, Sat(\Phi)) \text{ that is } \Pr(s_1, X\Phi) = \Pr(s_2, X\Phi).$$

Let $B = [s_1]_{\equiv_{\mathrm{CSL}_{\setminus X}}} = [s_2]_{\equiv_{\mathrm{CSL}_{\setminus X}}}$. First observe that $\mathbf{P}(s_1, B) = \mathbf{P}(s_2, B)$; otherwise, if, e.g., $\mathbf{P}(s_1, B) < \mathbf{P}(s_2, B)$ one would have $\Pr(s_1, \Diamond^{\leqslant t}\neg\Phi_B) < \Pr(s_2, \Diamond^{\leqslant t}\neg\Phi_B)$ for some sufficiently small t, contradicting $s_1 \equiv_{\mathrm{CSL}_{\setminus X}} s_2$. Distinguish:

- $\mathbf{P}(s_1, B) = \mathbf{P}(s_2, B) < 1$. As $s_1 \equiv_{\mathrm{CSL}_{\setminus X}} s_2$ and $\Phi_B \,\mathcal{U}\, \Phi$ is a $\mathrm{CSL}_{\setminus X}$-path formula: $\Pr(s_1, \Phi_B \,\mathcal{U}\, \Phi) = \Pr(s_2, \Phi_B \,\mathcal{U}\, \Phi)$. Using the same arguments as in the proof of Theorem 50 we obtain:

$$\Pr(s_i, \Phi_B \,\mathcal{U}\, \Phi) \;=\; \frac{\mathbf{P}(s_i, Sat(\Phi))}{1 - \mathbf{P}(s_i, B)}, \quad i = 1, 2.$$

  Since $\mathbf{P}(s_1, B) = \mathbf{P}(s_2, B)$, it follows $\mathbf{P}(s_1, Sat(\Phi)) = \mathbf{P}(s_2, Sat(\Phi))$.

- $\mathbf{P}(s_1, B) = \mathbf{P}(s_2, B) = 1$. As $Sat(\Phi)$ is the union of equivalence classes under $\equiv_{\mathrm{CSL}_{\setminus X}}$, the intersection with B is either empty or equals B. For $i = 1, 2$: $\mathbf{P}(s_i, Sat(\Phi)) = 1$ if $B \subseteq Sat(\Phi)$ and 0 if $B \cap Sat(\Phi) = \varnothing$. Hence, $\mathbf{P}(s_1, Sat(\Phi)) = \mathbf{P}(s_2, Sat(\Phi))$.

Thus, $s_1 \equiv_{\mathrm{PCTL}} s_2$. ■

Theorem 53. *For any CTMC: $\approx_c$ coincides with $\equiv_{CSL_{\setminus X}}$.*

Proof: We derive:

$$
\begin{array}{lll}
 & s_1 \approx_c^{\mathcal{C}} s_2 & \\
\text{iff} & s_1 \approx_c^{\mathit{unif}(\mathcal{C})} s_2 & \text{(by Prop. 33.3)} \\
\text{iff} & s_1 \sim_c^{\mathit{unif}(\mathcal{C})} s_2 & \text{(by Prop. 33.2)} \\
\text{iff} & s_1 \equiv_{\mathrm{CSL}}^{\mathit{unif}(\mathcal{C})} s_2 & \text{(by Theorem 49)} \\
\text{iff} & s_1 \equiv_{\mathrm{CSL}_{\setminus X}}^{\mathit{unif}(\mathcal{C})} s_2 & \text{(by Prop. 52)} \\
\text{iff} & s_1 \equiv_{\mathrm{CSL}_{\setminus X}}^{\mathcal{C}} s_2 & \text{(by Prop. 51)}
\end{array}
$$

■

Remark. The proof of the preservation property for $\mathrm{CSL}_{\setminus X}$ and $\approx_c$ seems to be simpler than for the discrete setting (cf. Theorem 50). An alternative proof of Theorem 50 could, however, be given which uses roughly the same arguments that we applied for the continuous case. For this, the concept of uniformization has to be adapted to FPSs (which amounts to just adding self-loops while keeping the relative probabilities for the original transitions unchanged) such that $\approx_d$ in the original FPS agrees with $\sim_d$ in the modified FPS. The remaining argumentation follows then as in the continuous case. ■

4.5. Safe and live fragments of PCTL and CSL

For the logical characterizations of the simulation relations, we distinguish between safety ("something bad never happens") and liveness ("something good will eventually happen") properties. In analogy to the universal and existential fragments of CTL, safe and live fragments of PCTL and CSL are defined as follows.

Safe and live PCTL

We consider only a restricted class of probability bounds in the probabilistic operator $\mathcal{P}$. The syntax of PCTL-safety formulae is as follows:

$$\Phi ::= \mathrm{tt} \;\Big|\; a \;\Big|\; \neg a \;\Big|\; \Phi \wedge \Phi \;\Big|\; \Phi \vee \Phi \;\Big|\; \mathcal{P}_{\geqslant p}(\widetilde{X}\,\Phi) \;\Big|\; \mathcal{P}_{\geqslant p}(\Phi\,\widetilde{\mathcal{U}}\,\Phi)$$

A typical safety property is $\mathcal{P}_{\geqslant 0.99}(\Box \neg\, \mathit{error})$ stating that with probability at least 0.99 the system will never be subject to an error. Using the duality of weak and strong until, $\mathcal{P}_{\leqslant 0.001}(\Diamond \mathit{error})$ is also a safety-formula and expresses that with probability at most 10^{-3} the system will eventually be subject to an error. Note that $\mathcal{P}_{\geqslant p}(\Psi\,\widetilde{\mathcal{U}}\,(\Phi \wedge \Psi)) = \mathcal{P}_{\leqslant 1-p}(\neg\,\Phi\,\mathcal{U}\,\neg\,\Psi)$; henceforth the latter formulae are also safety properties.

PCTL-liveness formulae are defined as follows:

$$\Phi ::= \mathrm{tt} \;\Big|\; a \;\Big|\; \neg a \;\Big|\; \Phi \wedge \Phi \;\Big|\; \Phi \vee \Phi \;\Big|\; \mathcal{P}_{\geqslant p}(X\Phi) \;\Big|\; \mathcal{P}_{\geqslant p}(\Phi\,\mathcal{U}\,\Phi)$$

Note that the weak next- and weak until-operator as allowed in safety-formulae, are replace by the traditional next- and until-operators. There is a duality between safety and liveness properties for PCTL, i.e., for any safety formula Φ there is a liveness property equivalent to $\neg\Phi$, and the same applies to liveness property Φ. This can easily be verified using structural induction on the syntax of safety PCTL-formulae.

Remark. In the context of safety formulae, next steps are viewed to be "dangerous" as they might violate safety. For instance, the safety formula $\mathcal{P}_{\geqslant 1-\varepsilon}(\widetilde{X}\,\mathit{safe})$ (which is equivalent to $\mathcal{P}_{\leqslant\varepsilon}(X\neg\,\mathit{safe})$) states that with sufficiently small probability the next state is unsafe. This is opposed to liveness properties such as $\mathcal{P}_{\geqslant 1-\varepsilon}(X\,\mathit{good})$ stating that with large probability a "good" next state occurs. ■

Safe and live CSL

The syntax of CSL-safety formulae is defined similar to that of safe PCTL:

$$\Phi ::= \mathrm{tt} \;\Big|\; a \;\Big|\; \neg a \;\Big|\; \Phi \wedge \Phi \;\Big|\; \Phi \vee \Phi \;\Big|\; \mathcal{P}_{\geqslant p}(\widetilde{X}^{\leqslant t}\Phi) \;\Big|\; \mathcal{P}_{\geqslant p}(\Phi\,\widetilde{\mathcal{U}}^{\leqslant t}\,\Phi)$$

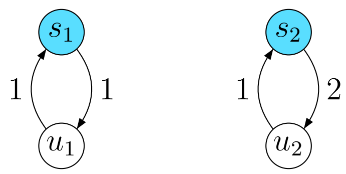
A typical safety property is $\mathcal{P}_{\geqslant 0.99}(\Box^{\leqslant 100}\neg\,\mathit{error})$ stating that with probability at least 0.99 the system will not exhibit an error for the next 100 time units.

CSL-liveness formulae are defined as follows:

$$\Phi ::= \mathrm{tt} \;\Big|\; a \;\Big|\; \neg a \;\Big|\; \Phi \wedge \Phi \;\Big|\; \Phi \vee \Phi \;\Big|\; \mathcal{P}_{\geqslant p}(X^{\leqslant t}\Phi) \;\Big|\; \mathcal{P}_{\geqslant p}(\Phi\,\mathcal{U}^{\leqslant t}\,\Phi)$$

There is a duality between safety and liveness properties for CSL as for PCTL.

Remark. The steady-state operator $\mathcal{S}_{\unlhd p}(\Phi)$ cannot be part of a CSL-fragment that enables a weak preservation result for $\precsim_c$. This is shown by the following example where we have $s_1 \precsim_c s_2$ and $u_1 \precsim_c u_2$.



The steady-state (or long-run) probabilities $\pi(s_1, s_1)$ and $\pi(s_1, u_1)$ are equal because the transitions $s_1 \to u_1$ and $u_1 \to s_1$ have the same speed. On the other hand, $s_2 \to u_2$ is twice as fast as $u_2 \to s_2$, hence, on the long run, the average time spent in u_2 is twice as that spent in s_2. Concretely,

$$\pi(s_1, s_1) = \pi(s_1, u_1) = \tfrac{1}{2} \text{ but } \pi(s_2, s_2) = \tfrac{1}{3} \text{ and } \pi(s_2, u_2) = \tfrac{2}{3}.$$

As a consequence,

$$s_1 \models \mathcal{S}_{\geqslant 0.5}(a), \text{ but } s_2 \not\models \mathcal{S}_{\geqslant 0.5}(a)$$

where we assume that $L(s_1) = L(s_2) = \{\, a \,\}$ and $L(u_1) = L(u_2) = \varnothing$. Vice versa,

$$s_2 \models \mathcal{S}_{\leqslant 0.5}(\neg a), \text{ while } s_1 \not\models \mathcal{S}_{\leqslant 0.5}(\neg a).$$

This example shows that there is no chance to find a comparison operator $\unlhd$ such that a preservation result for $\mathcal{S}$-formulae and $\precsim_c$ can be established. The fact that the steady-state operator is not compatible with our simulation relation can be viewed as a specific instance of the well-known phenomenon that CTMCs cannot be ordered according to their steady-state performance [59,16]. ■

4.6. Logical characterization of simulation

For DTMCs without absorbing states, $\precsim_d$ equals $\sim_d$ [45], and hence, equals $\equiv_{\text{PCTL}}$. For FPS where $\precsim_d$ is non-symmetric and strictly coarser than $\sim_d$, a logical characterization is obtained by considering a fragment of PCTL in the sense that $s \precsim_d s'$ iff all PCTL-safety properties that hold for s' also hold for s. In this sense, $\precsim_d$ can be read as: $s \precsim_d s'$ iff "s' is safer than s". For an action-labeled version of PCTL (in fact, a simpler modal logic with conjunction, disjunction and a next-step operator), such result was first presented by Desharnais *et al.* [24,26]. A similar result can be established for $\precsim_c$ and a safe fragment of CSL, as we will show below. The main results of this section are the weak preservation property for $\precsapprox$ stating that if $s \precsapprox s'$ then all PCTL$_{\setminus X}$-safety formulas that hold for state s' are also satisfied by s. A similar new result is obtained for the continuous case.

For convenience, we introduce the following notation: let $s \precsim^{\mathit{safe}}_{\text{PCTL}} s'$ if and only if for all PCTL-safety formulae Φ: $s' \models \Phi$ implies $s \models \Phi$. Likewise, $s \precsapprox^{\mathit{safe}}_{\text{PCTL}_{\setminus X}} s'$ if and only if this implication holds for all PCTL$_{\setminus X}$-safety formulae. Let $s \precsim^{\mathit{live}}_{\text{PCTL}} s'$ if and only if for all PCTL-liveness formulae Φ: $s \models \Phi$ implies $s' \models \Phi$. The preorder $\precsapprox^{\mathit{live}}_{\text{PCTL}_{\setminus X}}$ is defined similarly, and the same applies for the preorders corresponding to the safe and live fragments of CSL and CSL$_{\setminus X}$.

Theorem 54. *For any FPS: $\precsim_d$ coincides with $\precsim^{\mathit{safe}}_{PCTL}$ and with $\precsim^{\mathit{live}}_{PCTL}$.*

Proof: The equivalence of $\precsim^{\mathit{safe}}_{\text{PCTL}}$ and $\precsim^{\mathit{live}}_{\text{PCTL}}$ follows from the duality of safety and liveness formulae. We will now prove that $\precsim_d$ coincides with $\precsim^{\mathit{live}}_{\text{PCTL}}$.

1. ($\Rightarrow$). Let $s \precsim_d s'$. We prove that $s \precsim^{\mathit{live}}_{\text{PCTL}} s'$ by showing that the sets $\mathit{Sat}(\Phi)$ for PCTL-live formula Φ are upward-closed wrt. $\precsim_d$, i.e., $\mathit{Sat}(\Phi)$ equals the set of states that simulate some Φ-state:

$$\mathit{Sat}(\Phi) \;=\; \mathit{Sat}(\Phi)\uparrow \;=\; \{s \in S \mid s' \precsim_d s \text{ for some } s' \in \mathit{Sat}(\Phi)\}.$$

This is proven by structural induction on Φ. We only consider the until operator – the proofs for the other cases are similar and simpler – and show that for PCTL-live formulae Φ and Ψ with $\mathit{Sat}(\Phi) = \mathit{Sat}(\Phi)\uparrow$ and $\mathit{Sat}(\Psi) = \mathit{Sat}(\Psi)\uparrow$ then for all s, $s' \in S$:

$$s \precsim_d s' \;\Rightarrow\; \Pr(s, \Phi\,\mathcal{U}\,\Psi) \leqslant \Pr(s', \Phi\,\mathcal{U}\,\Psi).$$

From this it follows from the semantics of PCTL that

$$s \precsim_d s' \;\Rightarrow\; (s \models \mathcal{P}_{\geqslant p}(\Phi\,\mathcal{U}\,\Psi) \;\Rightarrow\; s' \models \mathcal{P}_{\geqslant p}(\Phi\,\mathcal{U}\,\Psi))$$

For convenience let $p(s)$ abbreviate $\Pr(s, \Phi\,\mathcal{U}\,\Psi)$. We have:

$$p(s) \;=\; \lim_{n\to\infty} p(s,n)$$

where $p(s,n)$ for natural n denotes the probability for a path fragment of length at most n which leads from s via Φ-states to a Ψ-state. Formally,

$$p(s,n) \;=\; \begin{cases} 1 & \text{if } \; s \models \Psi \\ \displaystyle\sum_{s' \in \mathit{Sat}(\Phi)\cup \mathit{Sat}(\Psi)} \mathbf{P}(s,s')\cdot p(s', n{-}1) & \text{if } \; s \models \Phi \wedge \neg\Psi \text{ and } n > 0 \\ 0 & \qquad \text{otherwise} \end{cases}$$

We now prove that $s \precsim_d s' \Rightarrow p(s,n) \leqslant p(s',n)$ for all n, and consequently, $p(s) \leqslant p(s')$. The proof proceeds by induction on n. For the base step $p(s,0) \in \{0,1\}$. $p(s,0) = 1$ if and only if $s \in Sat(\Psi)$, but as $Sat(\Psi)$ is upward-closed wrt. $\precsim_d$ and $s \precsim_d s'$ it follows $s' \in Sat(\Psi)$, and hence $p(s',0) = 1$. The case $p(s,0) = 0$ follows in a similar way. Distinguish two cases for the induction step. Let $n > 0$.

(a) $s' \models \Psi$. Then, $p(s',n) = 1 \geqslant p(s,n)$ for all n.

(b) $s' \not\models \Psi$. As $Sat(\Psi)$ is upward-closed, $s \not\models \Psi$. If $s \not\models \Phi$ then by definition of $p(s,n)$ we have $p(s,n) = 0 \leqslant p(s',n)$, for all n. The interesting case is when $s \models \Phi$, and as $Sat(\Phi)$ is upward-closed, $s' \models \Phi$. Let Δ be a weight function wrt. $\precsim_d$ for the distributions $s'' \mapsto \mathbf{P}(s,s'')$ and $s'' \mapsto \mathbf{P}(s',s'')$. As $Sat(\Phi)$ and $Sat(\Psi)$ are upward-closed and $\Delta(u_1,u_2) = 0$ if $u_1 \not\precsim_d u_2$ we have:

$$\Delta(u_1,u_2) = 0 \quad \text{if} \quad u_1 \in Sat(\Phi) \cup Sat(\Psi) \text{ and } u_2 \notin Sat(\Phi) \cup Sat(\Psi). \tag{7}$$

We now derive:

$$\begin{aligned}
& p(s,n{+}1) \\
= & \sum_{u_1 \in Sat(\Phi)\cup Sat(\Psi)} \mathbf{P}(s,u_1) \cdot p(u_1,n) && \text{by definition of } p(s,n) \\
= & \sum_{u_1 \in Sat(\Phi)\cup Sat(\Psi)} \sum_{u_2 \in S} \Delta(u_1,u_2) \cdot p(u_1,n) && \text{as } s \precsim_d s' \\
= & \sum_{\substack{u_1,u_2 \in Sat(\Phi)\cup Sat(\Psi) \\ u_1 \precsim_d u_2}} \Delta(u_1,u_2) \cdot p(u_1,n) && \text{by (7)} \\
\leqslant & \sum_{u_2 \in Sat(\Phi)\cup Sat(\Psi)} \sum_{u_1 \in Sat(\Phi)\cup Sat(\Psi)} \Delta(u_1,u_2) \cdot p(u_2,n) && \text{by induction hypothesis} \\
\leqslant & \textstyle\sum_{u_2 \in Sat(\Phi)\cup Sat(\Psi)} \sum_{u_1 \in S} \Delta(u_1,u_2) \cdot p(u_2,n) \\
= & \sum_{u_2 \in Sat(\Phi)\cup Sat(\Psi)} \mathbf{P}(s_2,u_2) \cdot p(u_2,n) && \text{as } \Delta \text{ is a weight function} \\
= & \; p(s_2,n{+}1) && \text{by definition of } p(s,n).
\end{aligned}$$

2. ($\Leftarrow$). We prove that $\precsim_{\text{PCTL}}^{live}$ is a weak probabilistic simulation. From the alternative characterization of $\precsim_d$ (cf. Prop. 20), it suffices to show that whenever $s \precsim_{\text{PCTL}}^{live} s'$ then $\mathbf{P}(s,C) \leqslant \mathbf{P}(s',C)$ for each $C \subseteq S$ which is upward-closed wrt. $\precsim_{\text{PCTL}}^{live}$. Let C be such an upward-closed set. For $u \in S \setminus C$ and $u' \in C$, there exists a PCTL-live formula $\Phi_{u',u}$ that distinguishes u and u' such that

$$u \notin Sat(\Phi_{u',u}) \quad \text{and} \quad u' \in Sat(\Phi_{u',u}).$$

Note that otherwise, we have $u' \precsim_{\text{PCTL}}^{live} u$, and hence, $u \in C$ (as C is upward-closed and $u' \in C$). Distinguish two cases.

(a) S is finite. Let

$$\Phi_{C,u} \;=\; \bigvee_{u' \in C} \Phi_{u',u}$$

for $u \in S \setminus C$. It directly follows

$$C \subseteq Sat(\Phi_{C,u}) \;\text{ and }\; u \notin Sat(\Phi_{C,u}).$$

Hence,

$$\Phi_C \;=\; \bigwedge_{u \in S \setminus C} \Phi_{C,u}$$

can be viewed as a master formula for C as $Sat(\Phi_C) = C$. Now consider the PCTL-live formulae $\Psi_p = \mathcal{P}_{\geqslant p}(X\,\Phi_C)$ where

$$p = \mathbf{P}(s, Sat(\Phi_C)) \;= \mathbf{P}(s, C).$$

Then, we have: $s \models \Psi_p$, and if $s \precsim^{live}_{\text{PCTL}} s'$, $s' \models \Psi_p$. Thus,

$$\mathbf{P}(s', C) \;=\; \mathbf{P}(s', Sat(\Phi_C)) \;\geqslant p \;=\; \mathbf{P}(s, C).$$

(b) S is countable infinite. As S is countable, we may use enumerations $u_1, u_2, \ldots$ of $S \setminus C$ and $u'_1, u'_2, \ldots$ of C and work with approximations of the above master formula (which cannot be defined as above because infinite disjunctions and conjunctions are not allowed in the syntax of PCTL). Let

$$\Phi^{(n)}_{C,u} \;=\; \bigvee_{1 \leqslant i \leqslant n} \Phi_{u'_i,u}.$$

Then,

$$\Phi^{(n,m)}_C \;=\; \bigwedge_{1 \leqslant j \leqslant m} \Phi^{(n)}_{C,u_j} \;\equiv\; \bigvee_{1 \leqslant i \leqslant n} \; \bigwedge_{1 \leqslant j \leqslant m} \Phi_{u'_i,u_j}$$

Let $C^{(n,m)} \;=\; Sat(\Phi^{(n,m)}_C)$. Then,

$$C \;=\; \bigcup_{n \geqslant 1} \bigcap_{m \geqslant 1} C^{(n,m)}.$$

As above, we obtain:

$$\mathbf{P}(s, C^{(n,m)}) \;\leqslant\; \mathbf{P}(s', C^{(n,m)}) \tag{8}$$

for all naturals n, $m \geqslant 1$. Moreover, we have:

$$\mathbf{P}(s, C) \;=\; \lim_{n \to \infty} \; \lim_{m \to \infty} \; \mathbf{P}(s, C^{(n,m)})$$

and similar for s'. By (8), we obtain $\mathbf{P}(s, C) \leqslant \mathbf{P}(s', C)$.

■

Theorem 55. *For any CTMC: $\precsim_c$ coincides with $\precsim^{safe}_{CSL}$ and with $\precsim^{live}_{CSL}$.*

Proof: To a large extent, the proof of this result goes along similar lines as the proof of Theorem 54. Due to the duality of CSL-safe and live-formulae, $\precsim^{safe}_{\text{CSL}}$ and with $\precsim^{live}_{\text{CSL}}$ coincide, and, hence, it suffices to show that $\precsim_c$ coincides with $\precsim^{live}_{\text{CSL}}$.

1. ($\Leftarrow$). Assume $s \precsim_{\mathrm{CSL}}^{\mathit{live}} s'$. With the same arguments as in the proof of Theorem 54 we obtain $L(s) = L(s')$ and $\mathbf{P}(s,C) \leqslant \mathbf{P}(s',C)$ for each upward-closed $C \subseteq S$ wrt. $\precsim_{\mathrm{CSL}}^{\mathit{live}}$. It remains (cf. Def. 22) to prove $E(s) \leqslant E(s')$. Consider the CSL-liveness formulae
$$\Phi \;=\; \mathcal{P}_{\geqslant p}(X^{\leqslant t}\mathrm{tt})$$
where $p = 1 - e^{-E(s)\cdot t}$. As $\Pr(s, X^{\leqslant t}\mathrm{tt}) = 1 - e^{-E(s)\cdot t}$ we have $s \models \Phi$, and as $s \precsim_{\mathrm{CSL}}^{\mathit{live}} s'$, $s' \models \Phi$. Therefore $1 - e^{-E(s')\cdot t} \;\geqslant\; p \;=\; 1 - e^{-E(s)\cdot t}$ which yields $E(s) \leqslant E(s')$. Thus $\precsim_{\mathrm{CSL}}^{\mathit{live}}$ is a strong simulation.
2. ($\Rightarrow$). As for Theorem 54, the crux of the proof is to show that for CSL-live formula Φ, $\mathit{Sat}(\Phi)$ is upward-closed wrt. $\precsim_c$. The main difference to the discrete setting is that $p(s,n)$ is replaced by $p(s,n,t)$, denoting the probability to fulfill the path formula $\Phi\,\mathcal{U}^{\leqslant t}\,\Psi$ via a path fragment of length at most n:
$$p(s,t,n) \;=\; \begin{cases} 1 & \text{if } \; s \models \Psi \\ \sum_{u\in S} \; \mathbf{R}(s,u)\cdot \int_0^t e^{-E(s)\cdot x}\cdot p(u, t{-}x, n{-}1)\; dx & \text{if } \; s \models \Phi \wedge \neg\Psi \text{ and } n > 0 \\ 0 & \text{otherwise} \end{cases}$$
The second clause is informally justified as follows. If s satisfies Φ and $\neg\Psi$, the probability of reaching a Ψ-state from s within t time units and n steps ($n > 0$) equals the probability of reaching some direct successor u of s in x time units ($x \leqslant t$), multiplied by the probability of reaching a Ψ-state from u in the remaining time $t{-}x$ (along a Φ-path) in $n{-}1$ steps.

   Let Φ and Ψ be CSL-formulae such that $\mathit{Sat}(\Phi)$ and $\mathit{Sat}(\Psi)$ are upward-closed wrt. $\precsim_c$. The interesting case is $s \models \Phi$ and $s \not\models \Psi$ (and the same for s'). As $s \precsim_c s'$, $E(s) \leqslant E(s')$. Now introduce a fresh state $\hat{s}$ with no incoming transitions, and with the same probabilistic structure as s, i.e., $\mathbf{P}(\hat{s},w) = \mathbf{P}(s,w)$ for all states w, but $E(\hat{s}) = E(s')$. $\hat{s}$ can be viewed as a "fast" copy of s. In particular, $p(s,t,n) \leqslant p(\hat{s},t,n)$. We now prove $p(\hat{s},t,n) \leqslant p(s',t,n)$ along similar lines as the proof of Theorem 54:
$$\begin{aligned}
& p(\hat{s}, t, n{+}1) \\
= & \int_0^t \sum_{u_1\in \mathit{Sat}(\Phi)\cup \mathit{Sat}(\Psi)} \mathbf{R}(\hat{s},u_1)\cdot e^{-E(s')\cdot x}\cdot p(u_1,t{-}x,n)\; dx \\
= & \int_0^t \sum_{u_1\in \mathit{Sat}(\Phi)\cup \mathit{Sat}(\Psi)} \sum_{u_2\in S} E(s')\cdot \Delta(u_1,u_2)\cdot e^{-E(s')\cdot x}\cdot p(u_1,t{-}x,n)\; dx \\
= & \int_0^t \sum_{\substack{u_1,u_2\in \mathit{Sat}(\Phi)\cup \mathit{Sat}(\Psi)\\ u_1 \precsim_d u_2}} E(s')\cdot \Delta(u_1,u_2)\cdot e^{-E(s')\cdot x}\cdot \underbrace{p(u_1,t{-}x,n)}_{\leqslant\, p(u_2,t-x,n),\ \text{by ind. hypo.}}\; dx \\
\leqslant & \int_0^t \sum_{u_2\in \mathit{Sat}(\Phi)\cup \mathit{Sat}(\Psi)} \sum_{u_1\in \mathit{Sat}(\Phi)\cup \mathit{Sat}(\Psi)} E(s')\cdot \Delta(u_1,u_2)\cdot e^{-E(s')\cdot x}\cdot p(u_2,t{-}x,n)\; dx \\
\leqslant & \int_0^t \sum_{u_2\in \mathit{Sat}(\Phi)\cup \mathit{Sat}(\Psi)} \sum_{u_1\in S} E(s')\cdot \Delta(u_1,u_2)\cdot e^{-E(s')\cdot x}\cdot p(u_2,t{-}x,n)\; dx
\end{aligned}$$

$$= \int_0^t \sum_{u_2 \in Sat(\Phi) \cup Sat(\Psi)} \mathbf{R}(s', u_2) \cdot e^{-E(s') \cdot x} \cdot p(u_2, t-x, n)\, dx$$

$$= p(s', t, n+1)$$

With $n \to \infty$ we obtain:

$$\Pr(s, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi) \;=\; \lim_{n\to\infty} p(s,t,n) \;\leqslant\; \lim_{n\to\infty} p(s',t,n) \;=\; \Pr(s', \Phi\,\mathcal{U}^{\leqslant t}\,\Psi)$$

■

The following two main results provide a relationship between the weak simulation pre-order and a pre-order on the safe (and live) fragments of $\text{PCTL}_{\setminus X}$ and $\text{CSL}_{\setminus X}$, respectively. As the proofs of these facts are non-trivial and proceed in several steps, we first give the result, present (as a remark) a first proof attempt, give a rough idea about the proof concept, and then the detailed proof. We start with the continuous case and then deal with the discrete case.

Theorem 56. *For any CTMC:* $\precapprox_c \subseteq \precapprox^{safe}_{CSL_{\setminus X}}$ *and* $\precapprox_c \subseteq \precapprox^{live}_{CSL_{\setminus X}}$.

Let $\mathcal{C} = (S, \mathbf{R}, L)$ be a CTMC. The aim is to show (as in the proof of Theorem 54) that $Sat(\Phi)$ for $\text{CSL}_{\setminus X}$-live formula Φ is upward-closed wrt. $\precapprox_c$. This is done by structural induction on the syntax of Φ. We concentrate on the time-bounded until operator, i.e., the proof obligation is to establish:

$$s \precapprox_c s' \text{ implies } \Pr(s, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi) \leqslant \Pr(s', \Phi\,\mathcal{U}^{\leqslant t}\,\Psi), \tag{9}$$

given that $Sat(\Phi)$ and $Sat(\Psi)$ are upward-closed wrt. $\precapprox_c$. As in the proofs of Theorems 54 and 55 the interesting case is $s, s' \in Sat(\Phi)$ and $s, s' \notin Sat(\Psi)$.

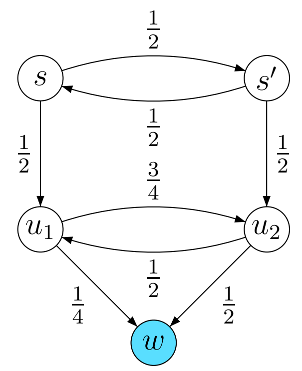
Remark. The initial proof idea for establishing (9) is to resort to the embedded uniformized CTMC of $\mathcal{C}$, using the result that:

$$\Pr^{\mathcal{C}}(s, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi) = e^{-E\cdot t} \cdot \sum_{k=0}^{\infty} \frac{(E\cdot t)^k}{k!} \cdot \Pr^{\mathcal{D}}(s, \Phi\,\mathcal{U}^{\leqslant k}\,\Psi), \tag{10}$$

where $\mathcal{D}$ is the embedded DTMC of $unif(\mathcal{C})$ and $\Phi\,\mathcal{U}^{\leqslant k}\,\Psi$ means that Ψ can be reached within at most k steps via a Φ-path (for natural k) [35]. The advantage of this approach would be that the remaining proof obligation:

$$s \precapprox_c s' \text{ implies } \Pr^{\mathcal{D}}(s, \Phi\,\mathcal{U}^{\leqslant k}\,\Psi) \;\leqslant\; \Pr^{\mathcal{D}}(s', \Phi\,\mathcal{U}^{\leqslant k}\,\Psi), \text{ for any } k \tag{11}$$

could be verified by considering the discrete-time behaviour of the CTMC only. Whereas the proof of equation (10) is rather straightforward, (11) turns out to be wrong. This is illustrated by the following (uniformized) CTMC $\mathcal{C}$:



where only the absorbing state is labeled by proposition b. It is not difficult to check that $s \precapprox_c s'$. Indeed it follows that

$$\Pr^{\mathcal{C}}(s, \Diamond^{\leqslant t} b) \leqslant \Pr^{\mathcal{C}}(s', \Diamond^{\leqslant t} b) \text{ for any real time instant } t.$$

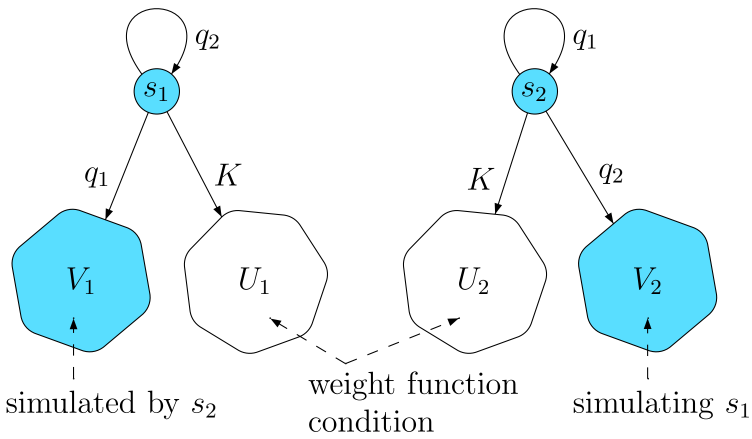
However, $\Pr^{emb(\mathcal{C})}(s, \Diamond^{\leqslant k} b) = \frac{7}{16} \not\leqslant \frac{3}{8} = \Pr^{emb(\mathcal{C})}(s', \Diamond^{\leqslant k} b)$ for $k = 3$. This contradicts (11). Thus, this initial proof attempt fails and we have to consider an alternative route. ■

We prove (9) therefore in a different way. In some sense, our argumentation is similar to the proof technique for the preservation for $\mathrm{CSL}_{\setminus X}$ and weak bisimulation (cf. Theorem 53). The rough idea is to replace $\mathcal{C}$ by a CTMC $\mathcal{C}'$ which results from $\mathcal{C}$ by adding self-loops.[6] Given two states s_1 and s_2 in $\mathcal{C}$ with $s_1 \precapprox_c s_2$ and a partitioning δ_1, U_1, V_1, δ_2, U_2, V_2, Δ as in Def. 39 we modify s_1 and s_2 by adding self-loops such that

- the probability q_2 for the additional self-loop at state s_1 equals the probability for s_2 to move to a V_2-state
- the probability q_1 for the additional self-loop at state s_2 equals the probability for s_1 to move to a V_1-state
- the probabilities for s_1 and s_2 to move to U_1 resp. U_2 are the same (i.e. $K_1 = K_2 = K$ for the modified states)
- the total rate to move from s_1 to a U_1-state is at most the total rate to move from s_2 to a U_2-state.

Thus, s_1 and s_2 are modified such that a CTMC is obtained with the following structure:

[6] This step can be seen as the analogue to the switch from $\mathcal{C}$ to *unif*$(\mathcal{C})$. However, the definition of $\mathcal{C}'$ is much more complicated than *unif*$(\mathcal{C})$.



The underlying idea behind this transformation is that the stutter-transitions $s_2 \to v_2 \in V_2$ can be mimicked by the additional self-loop $s_1 \to s_1$, and vice versa, the self-loop $s_2 \to s_2$ simulates the stutter-steps $s_1 \to v_1 \in V_1$. We then can continue similar to the proof of Theorem 55 and show by inductive arguments that

$$\Pr^{\mathcal{C}'}(s_1, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi) \quad \leqslant \quad \Pr^{\mathcal{C}'}(s_2, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi).$$

On the other hand, adding a self-loop (with arbitrary rate) does not change the weak bisimulation equivalence class, and hence, does not change the probabilities of the $\mathrm{CSL}_{\setminus X}$-path formulae (cf. Theorem 53):

$$\Pr^{\mathcal{C}}(s, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi) \quad = \quad \Pr^{\mathcal{C}'}(s, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi)$$

for all states s. Putting things together, we obtain

$$\Pr^{\mathcal{C}}(s_1, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi) \quad \leqslant \quad \Pr^{\mathcal{C}}(s_2, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi).$$

These are the underlying proof ideas. In fact, we have to work with several copies of the states and work with transitions leading from a copy of s_1 to several copies of s_1 (instead of simply adding self-loops). Before we present the details of these transformations, we make the following simplifying assumptions:

(A1) As $\mathrm{CSL}_{\setminus X}$-satisfaction on $\mathcal{C}$ and on *unif*$(\mathcal{C})$ agrees (cf. Prop. 51), we may assume that the exit rate of any state in $\mathcal{C}$ equals E. For the sake of simplicity let

$$E = E(s) = 1 \quad \text{for all states } s \in S$$

in the sequel. (In particular, $\mathcal{C}$ does not have absorbing states.)

(A2) For technical reasons, we assume that CTMC $\mathcal{C}$ does not have any self-loops, i.e., $\mathbf{R}(s,s) = 0$ for all states s. This assumption just simplifies the formulae for the rates in the modified CTMC $\mathcal{C}'$ and is not a real restriction: any self-loop $s \to s$ in $\mathcal{C}$ can be replaced by $s \to s'$ and $s' \to s$ where s' is a fresh copy of s. This transformation does not affect $[s]_{\sim_c}$.

(A3) For any pair $\langle s_1, s_2 \rangle$ of states in $\mathcal{C}$ with $s_1 \precsim_c s_2$, we fix functions $\delta_1 = \delta_1^{\langle s_1, s_2 \rangle}$, $\delta_2 = \delta_2^{\langle s_1, s_2 \rangle}$ and a weight function $\Delta = \Delta^{\langle s_1, s_2 \rangle}$ as in Def. 39. Furthermore, U_1, U_2, V_1, V_2, K_1, K_2 are as in Def. 39. In particular, we have:

$$K_1 \leqslant K_2$$

because $\mathcal{C}$ is uniformized.[7]

To simplify the formulae for the transition probabilities and rates in the modified CTMC $\mathcal{C}'$, we assume that δ_i is the characteristic function of U_i. In particular, $U_i \cap V_i = \varnothing$. Again, this is a harmless restriction because we may split any state $w \in U_i \cap V_i$ into two copies: one copy w_U belongs to U_i, the other copy w_V one to V_i. Then, the incoming transition $s_i \to w$ has to be split into the transitions $s_i \to w_U$ with rate $\delta_i(w) \cdot \mathbf{R}(s_i, w)$ and $s_i \to w_V$ with rate $(1 - \delta_i(w)) \cdot \mathbf{R}(s_i, w)$. This transformation does not affect $[s_i]_{\sim_c}$.

Let $\mathcal{C} = (S, \mathbf{R}, L)$ be the original CTMC as before. We replace $\mathcal{C}$ by a "state-wise" weakly bisimulation equivalent uniformized CTMC $\mathcal{C}' = (S', \mathbf{R}', L')$. The states of this transformed CTMC $\mathcal{C}'$ are of the form $\langle s_1, s_2, i \rangle$ with $i = 1, 2$ and $s_1 \precsim_c s_2$. Intuitively, the new state $\langle s_1, s_2, i \rangle$ is a copy of the original state s_i up to additional transitions inside $[s_i]_{\approx_c}$. For technical reasons, also the original states of $\mathcal{C}$ belong to $\mathcal{C}'$. Thus, we define the state space S' by:

$$S' \;=\; \{\langle s_1, s_2 \rangle \mid s_1, s_2 \in S.\, s_1 \;\precsim_c\; s_2\} \times \{\,1, 2\,\} \;\cup\; S$$

(where we assume that none of the states in S has the form $\langle s_1, s_2, i \rangle$ for i=1, 2). The labeling function L' in $\mathcal{C}'$ labels state $\langle s_1, s_2, 1 \rangle$ with the same atomic propositions as s_1, while the labeling of $\langle s_1, s_2, 2 \rangle$ agrees with the labeling of s_2:

$$L'(\langle s_1, s_2, i \rangle) = L(s_1) = L(s_2), \quad i = 1, 2.$$

The original states are unchanged, i.e., $L'(s) = L(s)$ for all states $s \in S$.

Below, the structure of the outgoing transitions from the states $\langle s_1, s_2, 1 \rangle$ and $\langle s_1, s_2, 2 \rangle$ is depicted:

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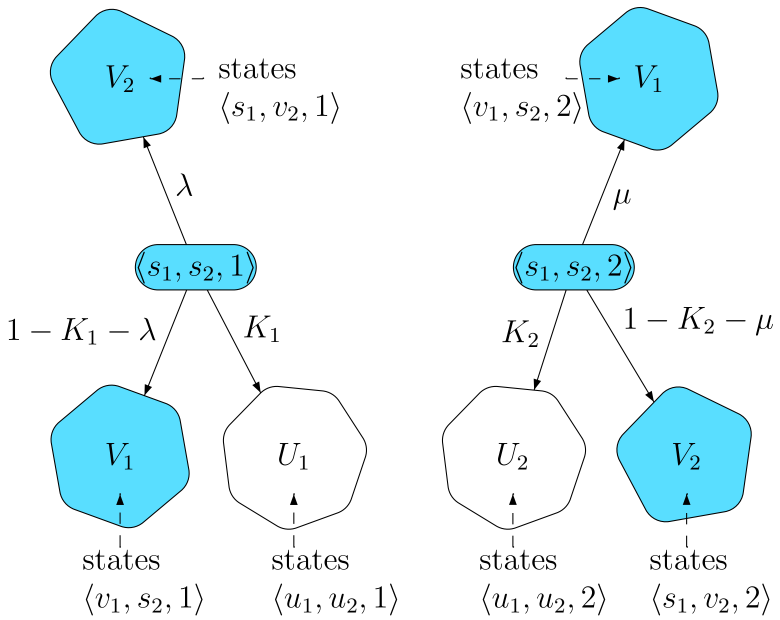
[7] The sets U_1, U_2, V_1, V_2 as well as K_1, K_2 depend on $\langle s_1, s_2 \rangle$. Thus, it would be more precise to write $U_1^{\langle s_1, s_2 \rangle}$, $U_2^{\langle s_1, s_2 \rangle}$, etc. Because in the sequel, we only use these components for a fixed pair $\langle s_1, s_2 \rangle$, we omit these parameters.



The total rate for the transitions of state $\langle s_1, s_2, 1\rangle$ to the auxiliary copies of s_1 for the V_2-states (i.e., the states $\langle s_1, v_2, 1\rangle$) is given by:

$$\lambda = \begin{cases} \left(\frac{1}{K_2} - 1\right) \cdot K_1 & \text{if } K_2 \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

The total rate for the transitions from $\langle s_1, s_2, 2\rangle$ to the states $\langle v_1, s_2, 2\rangle$ is defined as:

$$\mu = \begin{cases} \left(\frac{1}{K_1} - 1\right) \cdot K_2 & \text{if } K_1 \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Here, $K_i = K_i^{<s_1,s_2>}$ as in assumption (A3). Note that $\lambda = \mu = 0$ if $K_1 = 0$.

The rates of the original states $s \in S$ are as in $\mathcal{C}$, i.e., $\mathbf{R}'(s, w) = \mathbf{R}(s, w)$ for all s, $w \in S$ and $\mathbf{R}'(s, \langle w_1, w_2, i\rangle) = 0$ for all $s \in S$ and $\langle w_1, w_2, i\rangle \in S' \setminus S$. The rates of the outgoing transitions from states $\langle s_1, s_2, 1\rangle$ and $\langle s_1, s_2, 2\rangle$ are defined with the help of the components U_i, V_i, K_i, Δ (cf. assumption (A3)). The rates for state $\langle s_1, s_2, 2\rangle$ are defined as follows:

- For $K_1 = 0$ (i.e., $U_1 = \varnothing$), we depart from the informal explanations above and define $\langle s_1, s_2, 2\rangle$ to be a proper copy of s_2, i.e., for all $w \in S'$:

$$\mathbf{R}'(\langle s_1, s_2, 2\rangle, w) = \mathbf{R}'(s_2, w)$$

- For $K_1 > 0$ (i.e., $U_1 \neq \varnothing$) let $u_i \in U_i$, $v_i \in V_i$, $i = 1, 2$, and:

$$\begin{aligned} \mathbf{R}'\Big(\langle s_1, s_2, 2\rangle, \langle u_1, u_2, 2\rangle\Big) &= K_2 \cdot \Delta(u_1, u_2) \\ \mathbf{R}'\Big(\langle s_1, s_2, 2\rangle, \langle s_1, v_2, 2\rangle\Big) &= \mathbf{P}(s_2, v_2) \end{aligned}$$

If $K_1 = 1$ then $V_1 = \varnothing$ and there is no need to insert auxiliary transitions from state $\langle s_1, s_2, 2\rangle$. For $0 < K_1 < 1$, let:[8]

$$\mathbf{R}'(\langle s_1, s_2, 2\rangle, \langle v_1, s_2, 2\rangle) \;=\; \mu \cdot \frac{\mathbf{P}(s_1, v_1)}{1-K_1}, \text{ for } v_1 \in V_1$$

In all remaining cases, let $\mathbf{R}'(\langle s_1, s_2, 2\rangle, w) = 0$.

The rates for state $\langle s_1, s_2, 1\rangle$ are defined as follows:

- For $K_1 > 0$, $u_1 \in U_1$, $u_2 \in U_2$ and $v_1 \in V_1$ let:

$$\begin{aligned}\mathbf{R}'\Big(\langle s_1, s_2, 1\rangle, \langle u_1, u_2, 1\rangle\Big) \;&=\; K_1 \cdot \Delta(u_1, u_2)\\ \mathbf{R}'\Big(\langle s_1, s_2, 1\rangle, \langle v_1, s_2, 1\rangle\Big) \;&=\; \mathbf{P}(s_1, v_1)\end{aligned}$$

  If $K_1 > 0$ and $K_2 < 1$:

$$\mathbf{R}'(\langle s_1, s_2, 1\rangle, \langle s_1, v_2, 1\rangle) \;=\; \lambda \cdot \frac{\mathbf{P}(s_2, v_2)}{1-K_2}, \text{ for } v_2 \in V_2$$

  Let $\mathbf{R}'(\langle s_1, s_2, 1\rangle, w) = 0$ in all cases not mentioned so far. For $K_2 = 1$ we have $V_2 = \varnothing$, and hence, no auxiliary stutter transitions from $\langle s_1, s_2, 1\rangle$ are needed.

- If $K_1 = 0$, let

$$\mathbf{R}'\Big(\langle s_1, s_2, 1\rangle, \langle v_1, s_2, 1\rangle\Big) \;=\; \mathbf{P}(s_1, v_1)$$

  for all $v_1 \in V_1$ and $\mathbf{R}'(\langle s_1, s_2, 1\rangle, w) \;=\; 0$ for all other states w.

As $K_1 \leqslant K_2$, the cases $K_2 = 0 \,\wedge\, K_1 > 0$ and $K_2 < 1 \,\wedge\, K_1 = 1$ are impossible. This explains the asymmetry in the definition of the rate matrix of $\mathcal{C}'$.

The following two lemmas determine the exit rates in $\mathcal{C}'$, and the transition probabilities, respectively. **Lemma 57.** *The exit-rates of states $\langle s_1, s_2, 1\rangle$ and $\langle s_1, s_2, 2\rangle$ in $\mathcal{C}'$ are:*

$$E'(\langle s_1, s_2, 1\rangle) \;=\; 1+\lambda \;\leqslant\; 1+\mu \;=\; E'(\langle s_1, s_2, 2\rangle)$$

Proof: If $K_1 = 0$ then, by definition of λ and μ, $\lambda = \mu = 0$. In this case, the total rates of $\langle s_1, s_2, i\rangle$ agree: $E(s_1) = E(s_2) = 1$. (Recall that all states in $\mathcal{C}$ have the total rate $E = 1$.) Assume $K_1 > 0$. For $K_2 < 1$ we derive:

$$\begin{aligned}E'(\langle s_1, s_2, 1\rangle) \;&=\; \sum_{v_2 \in V_2} \lambda \cdot \frac{\mathbf{P}(s_2, v_2)}{1-K_2} \;+\; \underbrace{\sum_{u_1 \in U_1, u_2 \in U_2} K_1 \cdot \Delta(u_1, u_2)}_{=\mathbf{P}(s_1, U_1) = K_1} \;+\; \underbrace{\sum_{v_1 \in V_1} \mathbf{P}(s_1, v_1)}_{=\mathbf{P}(s_1, V_1) = 1-K_1}\\ &=\; \lambda \cdot \frac{1}{1-K_2} \cdot \underbrace{\sum_{v_2 \in V_2} \mathbf{P}(s_2, v_2)}_{=\mathbf{P}(s_2, V_2) = 1-K_2} \;+ K_1 \;+\; (1-K_1)\\ &=\; \lambda \cdot \frac{1}{1-K_2} \cdot (1-K_2) \;+\; 1\\ &=\; \lambda \;+\; 1.\end{aligned}$$

[8]Note that only the following formula has to be modified if $\mathcal{C}$ contains self-loops: the rate for the self-loop $s_1 \rightarrow s_1$ if $v_1 = s_1$ needs to be added.

For $K_2 = 1$ we immediately obtain that

$$E'(\langle s_1, s_2, 1\rangle) \;=\; \mathbf{P}(s_1, U_1) + \mathbf{P}(s_1, V_1) \;=\; 1 \;=\; 1 + \lambda$$

as $\lambda = 0$. Similarly, we get: $E'(\langle s_1, s_2, 2\rangle) \;=\; 1 + \mu$.

Because of the rate condition we have $K_1 \leqslant K_2$, and hence, $1/K_2 \leqslant 1/K_1$, if $K_1 > 0$. Therefore

$$\lambda \;=\; \Big(\underbrace{\frac{1}{K_2}}_{\leqslant \frac{1}{K_1}} - 1\Big) \cdot \underbrace{K_1}_{\leqslant K_2} \;\leqslant\; \Big(\frac{1}{K_1} - 1\Big) \cdot K_2 \;=\; \mu$$

■

We now show that there is a state-wise correspondence between the successors of $\langle s_1, s_2, 1\rangle$ and $\langle s_1, s_2, 2\rangle$.

Lemma 58. *For all states $\langle s_1, s_2, i\rangle$ and $\langle w_1, w_2, i\rangle$ with i=1, 2 in $\mathcal{C}'$ where $K_1 = K_1^{<s_1,s_2>} > 0$:*

$$\mathbf{P}'(\langle s_1, s_2, 1\rangle, \langle w_1, w_2, 1\rangle) \;=\; \mathbf{P}'(\langle s_1, s_2, 2\rangle, \langle w_1, w_2, 2\rangle)$$

Proof: By assumption $K_1 > 0$, and hence (as $K_1 \leqslant K_2$), $K_2 > 0$. We first consider the stutter-transitions to the V-states. The total probability for $\langle s_1, s_2, 1\rangle$ to move to $\langle v_1, s_2, 1\rangle$ is:

$$\mathbf{P}'(\langle s_1, s_2, 1\rangle, \langle v_1, s_2, 1\rangle) \;=\; \frac{\mathbf{P}(s_1, v_1)}{1 + \lambda} \;=\; \frac{\mathbf{P}(s_1, v_1)}{1 + (1/K_2 - 1)K_1}$$

$$=\; \frac{K_2 \cdot \mathbf{P}(s_1, v_1)}{K_2 + (1 - K_2)K_1} \;=\; \frac{K_2 \cdot \mathbf{P}(s_1, v_1)}{K_2 + K_1 - K_2 \cdot K_1}$$

This equals the probability for moving from $\langle s_1, s_2, 2\rangle$ to $\langle v_1, s_2, 2\rangle$, as for $0 < K_1 < 1$:

$$\mathbf{P}'(\langle s_1, s_2, 2\rangle, \langle v_1, s_2, 2\rangle) \;=\; \frac{\mathbf{P}(s_1, v_1)}{1 - K_1} \cdot \frac{\mu}{1 + \mu} \;=$$

$$\frac{\mathbf{P}(s_1, v_1)}{1 - K_1} \cdot \frac{(1 - K_1)K_2}{K_1 + (1 - K_1)K_2} \;=\; \frac{\mathbf{P}(s_1, v_1) \cdot K_2}{K_1 + (1 - K_1)K_2} \;=\; \frac{K_2 \cdot \mathbf{P}(s_1, v_1)}{K_1 + K_2 - K_1 \cdot K_2}$$

Note that the assumption $v_1 \in V_1$ implies $V_1 \neq \varnothing$, and hence, $K_1 < 1$. Similarly, the probability for the auxiliary transition from $\langle s_1, s_2, 1\rangle$ to $\langle s_1, v_2, 1\rangle$ coincides with the probability for $\langle s_1, s_2, 2\rangle$ to move to $\langle s_1, v_2, 2\rangle$. Thus, for all $v_1 \in V_1$ and $v_2 \in V_2$:

$$\mathbf{P}'(\langle s_1, s_2, 1\rangle, \langle v_1, s_2, 1\rangle) \;=\; \mathbf{P}'(\langle s_1, s_2, 2\rangle, \langle v_1, s_2, 2\rangle)$$

$$\mathbf{P}'(\langle s_1, s_2, 1\rangle, \langle s_1, v_2, 1\rangle) \;=\; \mathbf{P}'(\langle s_1, s_2, 2\rangle, \langle s_1, v_2, 2\rangle)$$

Now consider the "visible" transitions to the U-states. The probability for $\langle s_1, s_2, 1\rangle$ to move to state $\langle u_1, u_2, 1\rangle$ (where $u_i \in U_i$) is:

$$\mathbf{P}'(\langle s_1, s_2, 1\rangle, \langle u_1, u_2, 1\rangle) \;=\; \frac{K_1 \cdot \Delta(u_1, u_2)}{1 + \lambda} \;=\; \frac{K_1 \cdot \Delta(u_1, u_2)}{1 + (1/K_2 - 1)K_1}$$

$$=\; \frac{K_1 \cdot K_2 \cdot \Delta(u_1, u_2)}{K_2 + (1 - K_2)K_1} \;=\; \frac{K_1 \cdot K_2 \cdot \Delta(u_1, u_2)}{K_2 + K_1 - K_2 \cdot K_1}$$

The probability for $\langle s_1, s_2, 2\rangle$ to go to $\langle u_1, u_2, 2\rangle$ (where $u_1 \in U_1$ and $u_2 \in U_2$) is:

$$\mathbf{P}'(\langle s_1, s_2, 2\rangle, \langle u_1, u_2, 2\rangle) = \frac{K_2 \cdot \Delta(u_1, u_2)}{1+\mu} = \frac{K_2 \cdot \Delta(u_1, u_2)}{1+(1/K_1 - 1)K_2}$$

$$= \frac{K_1 \cdot K_2 \cdot \Delta(u_1, u_2)}{K_1 + (1-K_1)K_2} = \frac{K_1 \cdot K_2 \cdot \Delta(u_1, u_2)}{K_1 + K_2 - K_1 \cdot K_2}$$

So, $\mathbf{P}'(\langle s_1, s_2, 1\rangle, \langle u_1, u_2, 1\rangle) = \mathbf{P}'(\langle s_1, s_2, 2\rangle, \langle u_1, u_2, 2\rangle)$. Note that implicitly $U_1, U_2 \neq \varnothing$ as we assumed $u_1 \in U_1$ and $u_2 \in U_2$. Hence, $K_1 > 0$ and $K_2 > 0$. ■

According to the following result, the original CTMC $\mathcal{C}$ and its transformed variant $\mathcal{C}'$ are weak bisimilar ($\approx_c$):

Lemma 59. *For all s_1, s_2 in $\mathcal{C}$ with $s_1 \precsim_c s_2$: $s_i \approx_c \langle s_1, s_2, i\rangle$ for i=1, 2.* *Proof:* Let R be the coarsest equivalence on S' which identifies the states s_i and $\langle s_1, s_2, i\rangle$. We show that R is a weak bisimulation on $\mathcal{C}'$. (Recall that $\mathcal{C}$ is a sub-CTMC of $\mathcal{C}'$.)

The labeling condition is clear. It remains to show the rate condition. For this, it suffices to prove that for all equivalence classes $C \in S'/R$:

(I) If s_1, $s_2 \in S$ with $s_1 \notin C$ and $s_1 \precsim_c s_2$ then $\mathbf{R}'(s_1, C) = \mathbf{R}'(\langle s_1, s_2, 1\rangle, C)$.

(II) If s_1, $s_2 \in S$ with $s_2 \notin C$ and $s_1 \precsim_c s_2$ then $\mathbf{R}'(s_2, C) = \mathbf{R}'(\langle s_1, s_2, 2\rangle, C)$.

We provide the proof of (I). (II) can be shown with similar arguments. As $s_1 \notin C$ and as R identifies all states of the form $\langle s_1, w, 1\rangle$ with s_1, none of the states $\langle s_1, v_2, 1\rangle$ belongs to C. Hence,

$$\begin{aligned}
& \mathbf{R}'(\langle s_1, s_2, 1\rangle, C) \\
= & \sum_{\substack{u_1 \in U_1, u_2 \in U_2 \\ \langle u_1, u_2, 1\rangle \in C}} \mathbf{R}'(\langle s_1, s_2, 1\rangle, \langle u_1, u_2, 1\rangle) + \sum_{\substack{v_1 \in V_1 \\ \langle v_1, s_2, 1\rangle \in C}} \mathbf{R}'(\langle s_1, s_2, 1\rangle, \langle v_1, s_2, 1\rangle) \\
= & \sum_{\substack{u_1 \in U_1, u_2 \in U_2 \\ \langle u_1, u_2, 1\rangle \in C}} K_1 \cdot \Delta(u_1, u_2) + \sum_{\substack{v_1 \in V_1 \\ \langle v_1, s_2, 1\rangle \in C}} \mathbf{P}(s_1, v_1) \\
= & \sum_{\substack{u_1 \in U_1 \cap C \\ u_2 \in U_2}} K_1 \cdot \Delta(u_1, u_2) + \sum_{v_1 \in V_1 \cap C} \mathbf{P}(s_1, v_1) \\
= & \sum_{u_1 \in U_1 \cap C} \underbrace{K_1 \cdot \sum_{u_2 \in U_2} \Delta(u_1, u_2)}_{=\mathbf{P}(s_1, u_1) = \mathbf{R}(s_1, u_1)} + \underbrace{\mathbf{P}(s_1, V_1 \cap C)}_{=\mathbf{R}(s_1, V_1 \cap C)} \\
= & \mathbf{R}(s_1, U_1 \cap C) + \mathbf{R}(s_1, V_1 \cap C) \\
= & \mathbf{R}(s_1, C) = \mathbf{R}'(s_1, C)
\end{aligned}$$

Recall that $E(s_1) = 1$. Hence, $\mathbf{R}(s_1, w) = \mathbf{P}(s_1, w)$. Moreover, $\mathbf{R}(s_1, w) = \mathbf{R}'(s_1, w)$ for all states $w \in S$. ■

By the preservation result for $\mathrm{CSL}_{\setminus X}$ and $\approx_c$ (cf. Theorem 53), the transformation from $\mathcal{C}$ to $\mathcal{C}'$ leaves the probabilities for time-bounded until-formulae invariant:

Lemma 60. *For all s_1, s_2 in $\mathcal{C}$ with $s_1 \precapprox_c s_2$:*

$$\mathrm{Pr}^{\mathcal{C}}(s_i, \Phi\, \mathcal{U}^{\leqslant t}\, \Psi) \;=\; \mathrm{Pr}^{\mathcal{C}'}(\langle s_1, s_2, i\rangle, \Phi\, \mathcal{U}^{\leqslant t}\, \Psi)$$

.

Due to this result, it suffices to establish

$$\mathrm{Pr}^{\mathcal{C}'}(\langle s_1, s_2, 1\rangle, \Phi\, \mathcal{U}^{\leqslant t}\, \Psi) \;\leqslant\; \mathrm{Pr}^{\mathcal{C}'}(\langle s_1, s_2, 2\rangle, \Phi\, \mathcal{U}^{\leqslant t}\, \Psi).$$

in order to prove the obligation (9).

Remark. If $K_1^{<s_1,s_2>} > 0$ for all s_1, s_2 in $\mathcal{C}$ with $s_1 \precapprox_c s_2$, we have $\langle s_1, s_2, 1\rangle \precsim_c \langle s_1, s_2, 2\rangle$. This follows from the observation that the relation

$$R \;=\; \{(\langle s_1, s_2, 1\rangle, \langle s_1, s_2, 2\rangle) \mid s_1, s_2 \in S.\ s_1 \;\precapprox_c\; s_2\ \}$$

is a strong simulation for $\mathcal{C}'$ (provided that all K_i's are non-zero!). This can be seen as follows. The labeling condition is obvious. A weight function for $(\langle s_1, s_2, 1\rangle, \langle s_1, s_2, 2\rangle)$ is obtained by

$$\begin{aligned}\Delta(\langle w_1, w_2, 1\rangle, \langle w_1, w_2, 2\rangle) \quad &= \quad \mathbf{P}'(\langle s_1, s_2, 1\rangle, \langle w_1, w_2, 1\rangle) \\ &\overset{\text{Lemma 58}}{=} \quad \mathbf{P}'(\langle s_1, s_2, 2\rangle, \langle w_1, w_2, 2\rangle)\end{aligned}$$

The rate condition was shown in Lemma 57. Hence, in this particular case, we may apply the preservation result for CSL-liveness formulae and *strong* simulation (cf. Theorem 55) to obtain that

$$\mathrm{Pr}^{\mathcal{C}'}(\langle s_1, s_2, 1\rangle, \Phi\, \mathcal{U}^{\leqslant t}\, \Psi) \leqslant \mathrm{Pr}^{\mathcal{C}'}(\langle s_1, s_2, 2\rangle, \Phi\, \mathcal{U}^{\leqslant t}\, \Psi).$$

However, as we allow for $K_1^{<s_1,s_2>} = 0$, in general, state $\langle s_1, s_2, 1\rangle$ does *not* strongly simulate $\langle s_1, s_2, 2\rangle$ (we only have $\langle s_1, s_2, 1\rangle \precapprox_c \langle s_1, s_2, 2\rangle$). Thus, we cannot simply apply Theorem 55 to prove the following lemma. ■

Lemma 61. *For all s_1, s_2 in $\mathcal{C}$ with $s_1 \precapprox_c s_2$ and $CSL_{\setminus X}$-live formulae Φ and Ψ such that $Sat(\Phi)$ and $Sat(\Psi)$ are upward-closed wrt. $\precapprox_c$:*

$$\mathrm{Pr}^{\mathcal{C}'}(\langle s_1, s_2, 1\rangle, \Phi\, \mathcal{U}^{\leqslant t}\, \Psi) \quad\leqslant\quad \mathrm{Pr}^{\mathcal{C}'}(\langle s_1, s_2, 2\rangle, \Phi\, \mathcal{U}^{\leqslant t}\, \Psi)$$

Proof: In essence, our argumentation is similar to the proof of Theorem 55. However, there are some technical differences.

For s in $\mathcal{C}'$, let $p(s,t,n)$ denotes the probability to reach a Ψ-state via Φ-states within n ($n \geqslant 0$) steps and time-bound t from state s. And let

$$p(s,t,\infty) \;=\; \lim_{n\to\infty} p(s,t,n) \;=\; \mathrm{Pr}^{\mathcal{C}'}(s, \Phi\, \mathcal{U}^{\leqslant t}\, \Psi).$$

Instead of proving $p(\langle s_1, s_2, 1\rangle, t, n) \leqslant p(\langle s_1, s_2, 2\rangle, t, n)$ as in the proof of Theorem 55, we establish

$$p(\langle s_1, s_2, 1\rangle, t, n) \;\leqslant\; p(\langle s_1, s_2, 2\rangle, t, \infty) \tag{12}$$

for all states s_1, s_2 in the original CTMC $\mathcal{C}$ with $s_1 \precapprox_c s_2$. As in the proof of Theorem 55 the case $\langle s_1, s_2, i\rangle \in Sat(\Phi) \setminus Sat(\Psi)$ (for $i{=}1, 2$) is of interest. The proof is by induction on n. The basis of induction is clear, as

$$p(\langle s_1, s_2, 1\rangle, t, 0) \;=\; 0 \;\leqslant\; p(\langle s_1, s_2, 2\rangle, t, \infty).$$

Consider the induction step $n \Longrightarrow n{+}1$. We first consider the case where

$$K_1 = K_1^{<s_1,s_2>} > 0.$$

Similar to the argumentation in the proof of Theorem 55, we first replace the faster state $\langle s_1, s_2, 2\rangle$ by a slower copy $\langle s_1, s_2, 2, \mathit{slow}\rangle$ with total rate[9]

$$E'(\langle s_1, s_2, 2, \mathit{slow}\rangle) \;=\; E'(\langle s_1, s_2, 1\rangle) \;=\; 1+\lambda$$

and, for all states $w \in S'$,

$$\mathbf{P}'(\langle s_1, s_2, 2, \mathit{slow}\rangle, w) \;=\; \mathbf{P}'(\langle s_1, s_2, 2\rangle, w).$$

As state $\langle s_1, s_2, 2, \mathit{slow}\rangle$ is slower than $\langle s_1, s_2, 2\rangle$ (but has the same transition probabilities), we obtain:

$$p(\langle s_1, s_2, 2, \mathit{slow}\rangle, t, \infty) \;\leqslant\; p(\langle s_1, s_2, 2\rangle, t, \infty)$$

The induction hypothesis yields that

$$\begin{array}{lcl} p(\langle s_1, v_2, 1\rangle, y, n) & \leqslant & p(\langle s_1, v_2, 2\rangle, y, \infty) \\ p(\langle v_1, s_2, 1\rangle, y, n) & \leqslant & p(\langle v_1, s_2, 2\rangle, y, \infty) \\ p(\langle u_1, u_2, 1\rangle, y, n) & \leqslant & p(\langle u_1, u_2, 2\rangle, y, \infty) \end{array}$$

for any real number $y \geqslant 0$ and states $v_1 \in V_1$, $v_2 \in V_2$ and all states $u_1 \in U_1$, $u_2 \in U_2$ where $\Delta(u_1, u_2) > 0$. Hence, we get:

$$\begin{array}{cl} & p(\langle s_1, s_2, 2\rangle, t, \infty) \\ \geqslant & p(\langle s_1, s_2, 2, \mathit{slow}\rangle, t, \infty) \\ = & \displaystyle\sum_{w \in S'} \underbrace{E'(\langle s_1, s_2, 2, \mathit{slow}\rangle)}_{1+\lambda} \cdot \mathbf{P}'(\langle s_1, s_2, 2\rangle, w) \cdot \int_0^t e^{-(1+\lambda)x} \cdot p(w, t{-}x, \infty)\;\, dx \\ = & \displaystyle\sum_{v_1 \in V_1} (1+\lambda) \cdot \underbrace{\mathbf{P}'(\langle s_1, s_2, 2\rangle, \langle v_1, s_2, 2\rangle)}_{=\mathbf{P}'(\langle s_1,s_2,1\rangle,\langle v_1,s_2,1\rangle)} \cdot \int_0^t e^{-(1+\lambda)x} \cdot \underbrace{p(\langle v_1, s_2, 2\rangle, t{-}x, \infty)}_{\geqslant p(\langle v_1,s_2,1\rangle,t-x,n)}\;\, dx \\ & + \displaystyle\sum_{v_2 \in V_2} (1+\lambda) \cdot \underbrace{\mathbf{P}'(\langle s_1, s_2, 2\rangle, \langle s_1, v_2, 2\rangle)}_{=\mathbf{P}'(\langle s_1,s_2,1\rangle,\langle s_1,v_2,1\rangle)} \cdot \int_0^t e^{-(1+\lambda)x} \cdot \underbrace{p(\langle s_1, v_2, 2\rangle, t{-}x, \infty)}_{\geqslant p(\langle s_1,v_2,1\rangle,t-x,n)}\;\, dx \end{array}$$

[9]In the proof of Theorem 55 we did the converse and replaced the slower state by a faster copy, but this is not relevant.

$$
\begin{aligned}
&+ \sum_{\substack{u_2 \in U_2 \\ u_1 \in U_1}} (1+\lambda) \cdot \underbrace{\mathbf{P'}(\langle s_1, s_2, 2\rangle, \langle u_1, u_2, 2\rangle)}_{=\mathbf{P'}(\langle s_1,s_2,1\rangle,\langle u_1,u_2,1\rangle)} \cdot \int_0^t e^{-(1+\lambda)x} \cdot \underbrace{p(\langle u_1, u_2, 2\rangle, t{-}x, \infty)}_{\geqslant p(\langle u_1,u_2,1\rangle,t-x,n)} \; dx \\
\geqslant \; & \sum_{v_1 \in V_1} (1+\lambda) \cdot \mathbf{P'}(\langle s_1, s_2, 1\rangle, \langle v_1, s_2, 1\rangle) \cdot \int_0^t e^{-(1+\lambda)x} \cdot p(\langle v_1, s_2, 1\rangle, t{-}x, n) \; dx \\
&+ \sum_{v_2 \in V_2} (1+\lambda) \cdot \mathbf{P'}(\langle s_1, s_2, 1\rangle, \langle s_1, v_2, 1\rangle) \cdot \int_0^t e^{-(1+\lambda)x} \cdot p(\langle s_1, v_2, 1\rangle, t{-}x, n) \; dx \\
&+ \sum_{\substack{u_2 \in U_2 \\ u_1 \in U_1}} (1+\lambda) \cdot \mathbf{P'}(\langle s_1, s_2, 1\rangle, \langle u_1, u_2, 1\rangle) \cdot \int_0^t e^{-(1+\lambda)x} \cdot p(\langle u_1, u_2, 1\rangle, t{-}x, n) \; dx \\
= \; & p(\langle s_1, s_2, 1\rangle, t, n{+}1).
\end{aligned}
$$

It remains to discuss the case $K_1 = K_1^{\langle s_1, s_2\rangle} = 0$. Then, we have $\lambda = 0$, $U_1 = \varnothing$ and $\mathsf{Post}(s_1) = V_1$. Hence,

$$E'(\langle s_1, s_2, 1\rangle) \; = \; 1. \tag{13}$$

Moreover, we obtain by the induction hypothesis and by Lemma 60:

$$
\begin{aligned}
p(\langle v_1, s_2, 1\rangle, t, n) \; & \overset{\text{ind. hypo.}}{\leqslant} \; p(\langle v_1, s_2, 2\rangle, t, \infty) \\
& \overset{\text{Lemma 60}}{=} \; p(s_2, t, \infty)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
& p(\langle s_1, s_2, 1\rangle, t, n+1) \\
\overset{(13)}{=} \; & \int_0^t \sum_{v_1 \in V_1} \mathbf{P'}(\langle s_1, s_2, 1\rangle, \langle v_1, s_2, 1\rangle) \cdot e^{-x} \cdot \underbrace{p(\langle v_1, s_2, 1\rangle, t, n)}_{\leqslant \, p(s_2, t, \infty), \text{ see above}} \; dx \\
\leqslant \; & \int_0^t \sum_{v_1 \in V_1} \mathbf{P'}(\langle s_1, s_2, 1\rangle, \langle v_1, s_2, 1\rangle) \cdot e^{-x} \cdot p(s_2, t, \infty) \; dx \\
= \; & p(s_2, t, \infty) \cdot \underbrace{\sum_{v_1 \in V_1} \mathbf{P'}(\langle s_1, s_2, 1\rangle, \langle v_1, s_2, 1\rangle)}_{= 1, \text{ as } V_1 = \mathsf{Post}(s_1)} \cdot \underbrace{\int_0^t e^{-x} \; dx}_{= 1 - e^{-t}} \\
= \; & p(s_2, t, \infty) \cdot (1 - e^{-t}) \\
\leqslant \; & p(s_2, t, \infty) \\
\overset{\text{Lemma 60}}{=} \; & p(\langle s_1, s_2, 2\rangle, t, \infty)
\end{aligned}
$$

With $n \to \infty$ in (12) we get the desired result. ■

Combining Lemma 61 and Lemma 60 yields the claim (9):

Lemma 62. *Let* Φ *and* Ψ *be* $CSL_{\setminus X}$*-formulae such that* $Sat(\Phi)$ *and* $Sat(\Psi)$ *are upward-closed wrt.* $\precsim_c$. *Then, for all* s_1 *and* s_2 *in* $\mathcal{C}$*:*

$$s_1 \precsim_c s_2 \quad \textit{implies} \quad \Pr(s_1, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi) \leqslant \Pr(s_2, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi).$$

Proof: Using the results above and defined transformations we derive:

$$\begin{array}{ll} & \Pr^{\mathcal{C}}(s_1, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi) \\ \overset{\text{Lemma 60}}{=} & \Pr^{\mathcal{C}'}(\langle s_1, s_2, 1\rangle, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi) \\ \overset{\text{Lemma 61}}{\leqslant} & \Pr^{\mathcal{C}'}(\langle s_1, s_2, 2\rangle, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi) \\ \overset{\text{Lemma 60}}{=} & \Pr^{\mathcal{C}}(s_2, \Phi\,\mathcal{U}^{\leqslant t}\,\Psi) \end{array}$$

■

Lemma 62 completes the proof of Theorem 56.

Theorem 63. *For any FPS:* $\precsim_d \subseteq \precsim^{safe}_{PCTL_{\setminus X}}$ *and* $\precsim_d \subseteq \precsim^{live}_{PCTL_{\setminus X}}$. *Proof: (Sketch).* As for the continuous case, it suffices to show for s_1, s_2 in FPS $\mathcal{D}$:

$$s_1 \precsim_d s_2 \text{ implies } \Pr(s_1, \Phi\,\mathcal{U}\,\Psi) \leqslant \Pr(s_2, \Phi\,\mathcal{U}\,\Psi),$$

provided that Φ and Ψ are $\mathrm{PCTL}_{\setminus X}$-formulae with upward-closed satisfaction sets wrt. $\precsim_d$.

Note that the approach for proving the correspondence between $\precsim_d$ and $\precsim^{live}_{\mathrm{PCTL}}$ (cf. Theorem 54) does not work as – in analogy to Remark 4.6 – it is possible that

$$\text{if } s_1 \precsim_d s_2 \text{ then } p(s_1, n) > p(s_2, n)$$

where $p(s, n)$ denotes the probability for paths of length at most n starting in s that fulfill $\Phi\,\mathcal{U}\,\Psi$. Instead, we use an argument similar to that for establishing the relation between $\precsim_c$ and $\precsim^{live}_{\mathrm{CSL}_{\setminus X}}$. More precisely, we modify $\mathcal{D} = (S, \mathbf{P}, L)$ into the FPS $\mathcal{D}' = (S', \mathbf{P}', L')$ that is "state-wise" weakly bisimilar to $\mathcal{D}$ such that for the copies s_1', s_2' of the states s_1 and s_2 in $\mathcal{D}$:

$$s_1 \precsim_d s_2 \quad \text{implies} \quad p^{\mathcal{D}'}(s_1', n) \leqslant p^{\mathcal{D}'}(s_2', n).$$

The transformation from $\mathcal{D}$ into $\mathcal{D}'$ is similar to the transformation for CTMCs used before. Let

$$S' = \{\, \langle s_1, s_2, i\rangle \;:\; s_1, s_2 \in S,\; s_1 \precsim_d s_2, \,\} \times \{1, 2\} \;\cup\; S$$

where $\langle s_1, s_2, i\rangle$ can be viewed as a copy of s_i. L' is defined as in the continuous case, i.e., $L'(\langle s_1, s_2, i\rangle) = L(s_i)$. The probability matrix $\mathbf{P}'$ of $\mathcal{D}'$ is obtained as follows. Let $s_1, s_2 \in \mathcal{D}$ with $s_1 \precsim_d s_2$. Assume that U_i, V_i, K_i, Δ are the components as in Def. 34 with $R = \precsim_d$. For $K_1 = 0$, all successors of s_1 belong to V_1. Hence, all states in $\mathsf{Post}(s_1)$ are simulated by s_2. In this case, no real modification is needed and we put

$$\mathbf{P}'(\langle s_1, s_2, 1\rangle, \langle v_1, s_2, 1\rangle) = \mathbf{P}(s_1, v_1) \text{ and } \mathbf{P}'(\langle s_1, s_2, 2\rangle, \langle s_1, w, 2\rangle) = \mathbf{P}(s_2, w)$$

for all states $v_1 \in V_1$ and $w \in \mathsf{Post}(s_2)$ and $\mathbf{P}'(\langle s_1, s_2, i\rangle, \cdot) = 0$ in the remaining cases. The definition for $K_2 = 0$ is similar and omitted here.

Now consider $K_1 > 0$ and $K_2 > 0$. As before, to simplify matters, let δ_i be the characteristic function of U_i (i.e., any successor state of s_i either belongs to U_i or to V_i). Let

$$H \;=\; (1-K_1)\cdot\frac{K_2}{K_1} \quad\text{and}\quad M \;=\; (1-K_2)\cdot\frac{K_1}{K_2}$$

and for $v_2 \in V_2$, $v_1 \in V_1$ and $u_1 \in U_1$, $u_2 \in U_2$:

$$\begin{aligned}
\mathbf{P}'(\langle s_1, s_2, 1\rangle, \langle v_1, s_2, 1\rangle) &= \frac{\mathbf{P}(s_1, v_1)}{1+M}\\
\mathbf{P}'(\langle s_1, s_2, 1\rangle, \langle u_1, u_2, 1\rangle) &= K_1\cdot\frac{\Delta(u_1, u_2)}{1+M}\\
\mathbf{P}'(\langle s_1, s_2, 1\rangle, \langle s_1, v_2, 1\rangle) &= \frac{M}{1+M}\cdot\frac{\mathbf{P}(s_2, v_2)}{1-K_2}
\end{aligned}$$

The transition probabilities for state $\langle s_1, s_2, 2\rangle$ are defined similarly. Then,

$$\mathbf{P}'(\langle s_1, s_2, 1\rangle, \perp) \;=\; \frac{\mathbf{P}(s_1, \perp)}{1+M} \quad\text{and}\quad \mathbf{P}'(\langle s_1, s_2, 2\rangle, \perp) \;=\; \frac{\mathbf{P}(s_2, \perp)}{1+H}$$

We now have:

$$\begin{aligned}
\mathbf{P}'(\langle s_1, s_2, 1\rangle, \langle s_1, v_2, 1\rangle) &= \mathbf{P}'(\langle s_1, s_2, 2\rangle, \langle s_1, v_2, 2\rangle)\\
\mathbf{P}'(\langle s_1, s_2, 1\rangle, \langle v_1, s_2, 1\rangle) &= \mathbf{P}'(\langle s_1, s_2, 2\rangle, \langle v_1, s_2, 2\rangle)\\
\mathbf{P}'(\langle s_1, s_2, 1\rangle, \langle u_1, u_2, 1\rangle) &= \mathbf{P}'(\langle s_1, s_2, 2\rangle, \langle u_1, u_2, 2\rangle)
\end{aligned}$$

Moreover, state s_i is weakly bisimilar to state $\langle s_1, s_2, i\rangle$. Hence, by Theorem 50:

$$\Pr^{\mathcal{D}}(s_i, \Phi\,\mathcal{U}\,\Psi) \;=\; \Pr^{\mathcal{D}'}(\langle s_1, s_2, i\rangle, \Phi\,\mathcal{U}\,\Psi)$$

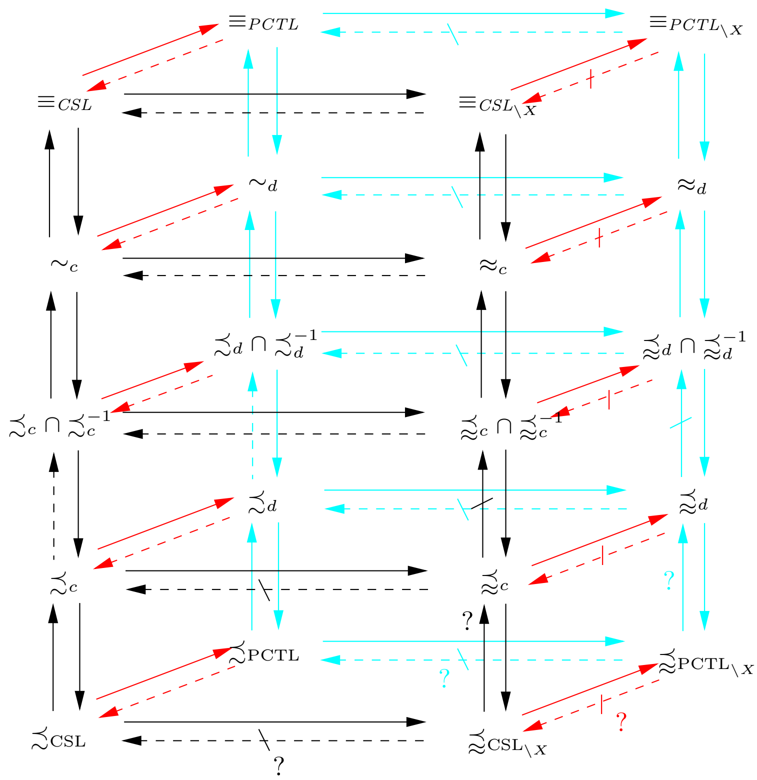
The rest of the argument is similar to the the proof of Theorem 56 and is omitted here. ■

5. Summary and conclusions

This section summarizes the main results in this paper and concludes.

5.1. The branching-time spectrum

Summarizing the results obtained and summarized in this paper yields the 3-dimensional spectrum of branching-time relations for Markov chains as depicted as follows:



All strong bisimulation relations are clearly contained within their weak variants, i.e., $\sim_d \subseteq \approx_d$ and $\sim_c \subseteq \approx_c$. The plane in the "front" (black arrows) represents the continuous-time setting, whereas the plane in the "back" (light blue or gray arrows) represents the discrete-time setting. Arrows connecting the two planes (red or dark gray) relate CTMCs and their embedded DTMCs. $R \longrightarrow R'$ means that R is finer than R', while $R \not\longrightarrow R'$ means that R is not finer than R'. The dashed arrows in the continuous setting refer to uniformized CTMCs, i.e., if there is a dashed arrow from R to R', R is finer than R' for uniformized CTMCs. In the discrete-time setting the dashed arrows refer to DTMCs without absorbing states. Note that these models are obtained as embeddings of uniformized CTMCs (except for the pathological CTMC where all exit rates are 0, in which case all relations in the picture agree). If a solid arrow is labeled with a question mark, we claim the result, but have no proof (yet). For negated dashed arrows with a question mark, we claim that the implication does not hold even for uniformized CTMCs (DTMCs without absorbing states). The only difference between the discrete and continuous setting is that weak and strong bisimulation equivalence agree for uniformized CTMCs, but not for DTMCs without absorbing states.

Remark. The weak bisimulation proposed in [3] is strictly coarser than $\approx_d$, and thus does not preserve $\equiv_{\mathrm{PCTL}_{\setminus X}}$. The ordinary, non-probabilistic branching-time spectrum is more diverse, because there are many different weak bisimulation-style equivalences [30]. In the setting considered here, the spectrum spanned by Milner-style observational equivalence and branching bisimulation equivalence collapses to a single "weak bisimulation equivalence" [9]. Another difference is that for ordinary transition systems, simulation equivalence is strictly coarser than bisimulation equivalence. Further, in this non-probabilistic setting weak relations have to be augmented with aspects of divergence to obtain a logical characterization by $\mathrm{CTL}_{\setminus X}$ [21]. In the probabilistic setting, divergence occurs with probability 0 or 1, and does not need any distinguished treatment.

Decision algorithms

For the sake of completeness, we briefly summarize the various decision algorithms that exist for the (bi)simulation relations considered here. Checking strong bisimulation on Markov chains can be done in time $\mathcal{O}(m\cdot \log n)$, where n is the number of states and m is the number of transitions [22]. This algorithm can also be employed for $\approx_c$. In the discrete-time case, checking $\sim_d$ takes $\mathcal{O}(m\cdot \log n)$ time [40], whereas $\approx_d$ take $\mathcal{O}(n^3)$ time [9]. The computation of $\precsim_d$ can be reduced to a maximum flow problem [7] and has a worst case time complexity of $\mathcal{O}((m\cdot n^6 + m^2\cdot n^3)/\log n)$. The same technique can be applied for computing $\precsim_c$. A polynomial-time algorithm for computing $\precapprox_c$ (and $\precapprox_d$) of a finite-state Markov chain was recently presented in [10]. The crux of this algorithm is to consider the check whether a state weakly simulates another one as a linear programming problem.

5.2. Concluding remarks

This paper has explored the spectrum of strong and weak (bi)simulation relations for countable fully probabilistic systems as well as continuous-time Markov chains. Based on a cascade of definitions in a uniform style, we have studied strong and weak (bi)simulations, and have provided logical characterizations in terms of fragments of PCTL and CSL. The definitions of the (bi)simulation relations have three ingredients: (1) a condition on the labeling of states with atomic propositions, (2) a time-abstract condition on the probabilistic behaviour, and (3) a model-dependent condition: a rate condition for CTMCs (on the exit rates in the strong case, and on the total rates of "visible" moves in the weak case), and a reachability condition on the "visible" moves in the weak FPS case. The strong FPS case does not require a third condition.

As the rate conditions imply the corresponding reachability condition, the "continuous" relations are finer than their "discrete" counterparts, and the continuous-time setting excludes the possibility to abstract from stuttering occurring with probability one.[10] While weak bisimulation in CTMCs (and FPSs) is a rather fine notion, it is the best abstraction preserving all properties that can be specified in CSL (PCTL) without next.

Issues for future work are the extension of this comparative semantics study towards models that exhibit both non-determinism and probabilities. As the models (and the (bi)simulation relations) in this setting are more diverse, this is non-trivial. Initial attempts towards such comparative studies can be found in [55] that compare simple probabilistic automata and alternating probabilistic transition systems. Another topic for future work is to complete the branching-time spectrum for Markov chain by proving the following conjectures: $\precapprox_d$ coincides with $\precapprox^{safe}_{\mathrm{PCTL}_{\setminus X}}$ and $\precapprox^{live}_{\mathrm{PCTL}_{\setminus X}}$, and $\precapprox_c$ coincides with $\precapprox^{safe}_{\mathrm{CSL}_{\setminus X}}$ and $\precapprox^{live}_{\mathrm{CSL}_{\setminus X}}$.

[10]In process-algebraic terminology, the reachability condition guarantees the law $\tau.P = P$ for FPS. This law cannot hold for CTMCs due to the advance of time while stuttering (performing τ).

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