Symmetry Reduction for Stochastic Hybrid Systems

Manuela L. Bujorianu and Joost-Pieter Katoen

Abstract— This paper is focused on adapting symmetry reduction, a technique that is highly successful in traditional model checking, to stochastic hybrid systems. To that end, we first show that performability analysis of stochastic hybrid systems can be reduced to a stochastic reachability analysis (SRA). Then, we generalize the notion of symmetry reduction as recently proposed for probabilistic model checking, to continuous probabilistic systems. We provide a rigorous mathematical foundation for the reduction technique in the continuous case and also investigate its observability perspective.

Keywords: Markov models, symmetries, transformation group, abstractions, reachability, probabilistic model checking.

I. INTRODUCTION

Symmetry reduction is a well-investigated technique for combatting the impact of state-explosion in temporal logic model checking (see [16], [12] and the references therein). It is a method to exploit the occurrence of replication in a model. This method has been applied mainly to models of concurrent systems of processes, such as communication and memory consistency protocols. Symmetry reduction gives the possibility to verify a model over a reduced quotient model, which is not only much smaller, but also bisimulation-equivalent to the original.

In the continuous setting, symmetry reduction techniques appear in different contexts. The collection of the planar motions that keep a geometric figure invariant form a group, called the symmetry group of the figure (rectangle, triangle, circle). It gives a measure for the symmetry degree of the figure, and it might help to reconstitute it from its parts. For an algebraic equation, a symmetry group is composed by the base space transformations that permute solutions. In the case of ordinary differential equations (ODE), all the special techniques for solving certain classes of ODE have their origin in a general method related to the existence of a continuous invariance group for these ODE (Lie theory [19]). For deterministic hybrid systems, a unifying framework in which to carry out the hybrid geometric reduction of hybrid systems, generalizing classical reduction to a hybrid setting has been developed in [1].

In the stochastic continuous case, symmetry features have been also employed in different frameworks. The symmetries of the Laplacian on the Euclidean space are of great help for studying properties of the Brownian motion. The diffusion

M.L.. Bujorianu is with Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente, 7500 AE Enschede, The Netherlands l.m.bujorianu@cs.utwente.nl

J.-P. Katoen is with RWTH Aachen University, Software Modelling and Verification Group, D-52056 Aachen, Germany katoen@informatik.rwth-aachen.de

processes having the maximal symmetry properties are characterised in [18].

In this paper, we generalize the symmetry reduction techniques as recently proposed for probabilistic model checking, to continuous probabilistic systems (briefly presented in Section IV). The main purpose of our investigations is to apply these techniques to stochastic hybrid systems [7], [10]. For continuous time/space Markov processes, when we generalise the symmetry reduction technique from [16], [12], we obtain nothing else, but the space reduction using invariance transformation groups beautifully exposed by Dynkin, E.B in [13], Ch. 10 (see content of Section V). The main difficulty in applying such a technique to stochastic hybrid systems is to find out the appropriate invariance transformations that act uniformly on the domains of different discrete modes (with corresponding diffusion processes and guards), and is compatible with the jumping part. This jumping part is given by the discrete transitions between modes and is governed by some rates and reset maps. To overcome this problem, we propose a novel approach for the symmetry reduction of the state space of a Markov process considering transformation groups that preserve 'observations' over the trajectories. We provide a rigorous mathematical foundation for this reduction technique and also prove that the reduced quotient model is bisimulation-equivalent to the original model (Section VI). Section VII is dedicated to applying these techniques to stochastic hybrid systems.

II. PROBABILISTIC MODELS

A probabilistic model is a transition system with the state space X, whose behaviour is specified not by a transition relation on X, but a transition function. The most known probabilistic models are: discrete-time Markov chains (DTMC), continuous-time Markov chains (CTMC), and Markov decision processes (MDP).

A. Discrete/Continuous-time Markov chains

DTMCs are defined by a function $P : X \times X \to [0, 1]$ satisfying $\sum_{x' \in X} P(x, x') = 1$ for each $x \in X$. This function is known as the *transition probability matrix*, gives the probability $P(x, x')$ of making a transition from each state x to any other state x'.

CTMCs are defined by a transition rate matrix $R : X \times X \to \mathbb{R}_+$ giving the rate $R(x, x')$ at which transitions between state pairs (x, x') occur. This rate is interpreted as the parameter of a negative exponential distribution, resulting in a dense model of time. If a CTMC is defined on a denumerable state space X and with the stochastic transition matrix $P(t) = (p_{xy}(t))$, where x and y range over X. Let

us denote by $Q = (q_{xy})$ the right-hand derivative at $t = 0$ of $P(t)$, i.e. the generator matrix of the chain. The entries of the infinitesimal generator matrix Q are the rates at which the process jumps from state to state.

B. Continuous time/space Markov processes

The stochastic processes we consider here are randomized systems with a continuous state space, where the "noise" can be measured using transition probability measures. Markov processes form a subclass of stochastic systems for which, at any stage, future evolutions are conditioned only by the present state (in other words, they do not depend on the past). State Space The state space is denoted by X. The basic assumption is that one can reason about state change using probabilities. Then the state space should be a measurable space. Suppose that X is a Polish or analytic space[1]. We consider X equipped with its Borel σ-algebra $\mathcal{B}$ (i.e. the σ-algebra generated by all open sets). The set of all bounded measurable numerical functions on X is denoted by $\mathbf{B}(X)$. This set can be thought of as an additive monoid $S = (\mathbf{B}(X), +, 0)$. These functions can be thought as abstract states (configurations) of the given system or, some formulas in an appropriate logic.

Sample Probability Space A probability space $(\Omega, \mathcal{F}, P)$ is fixed and all $X-$valued random variables are defined on this probability space. The trajectories in the state space are modelled by a family of random variables (x_t) where t denotes the time. The reasoning about state change is carried out by a family of probabilities P_x one for each state $x \in X$. For Markov processes, for each state x, the probability $P_x(x_t \in A)$ to reach a given measurable set of states $A \subset X$ starting from x describes the system evolution. Technically, with any state $x \in X$ we can associate a natural probability space $(\Omega, \mathcal{F}, P_x)$ where P_x is a probability measure such that $P_x(x_t \in A)$ is $\mathcal{B}$-measurable in $x \in X$, for each $t \in [0, \infty)$ and $A \in \mathcal{B}$, and its initial probability distribution is $P_x(x_0 = x) = 1$. An extra point ∂ (the cemetery or deadlock point) is added to X as an isolated point, $X_\partial = X \cup \{\partial\}$. Let $\mathcal{B}(X_\partial)$ be the Borel σ-algebra of X_∂. The existence of ∂ is assumed to have a probabilistic interpretation of $P_x(x_t \in X) < 1$, i.e. ∂ is the state where the process resides when it 'dies'.

Strong Markov Property Formally, let $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P, P_x)$ be a strong Markov process with the state space X, and with underlying probability space $(\Omega, \mathcal{F}, P)$. X is equipped with its Borel σ-algebra (generated by the open sets), denoted by $\mathcal{B}(X)$. $\mathcal{F}_t$ describes the history of the process up to the time t ($\mathcal{F}_t$ is the σ-algebra generated by the random variables x_s, $s \leq t$). Strong Markov property means that the Markov property is still true w.r.t. the stopping times of the process M. Recall that a $[0, \infty]$-valued function τ on Ω is called a *stopping time* if it is measurable w.r.t. the history of the process. The trajectories of M are modelled by a family of X-valued random

variables (x_t), which, as functions of time, can have some continuity properties (as the càdlàg[2] property, i.e. right continuous with left limits). The 'termination time' $\zeta(\omega)$ is the random time when the process M escapes to and is trapped at ∂.

Transition Function A transition function $p_t(x, \Gamma)$ is a *transition probability function* for a time homogeneous Markov process if $P\{x_{t+s} \in \Gamma | \mathcal{F}_t\} = p_s(x_t, \Gamma)$, for all $s, t \geq 0$ and $\Gamma \in \mathcal{B}(X)$. The relation between the transition probabilities and the Wiener probabilities is given by $p_t(x, \Gamma) = P_x(x_t \in \Gamma)$, for all $t \geq 0$ and $\Gamma \in \mathcal{B}(X)$.

Semigroup of operators The foundation of the connections between Markov processes and analysis is given by the concept of the shift of a function defined on the state space X. Let us consider any nonnegative measurable function $\tau : \Omega \rightarrow \mathbb{R}_+$ (positive random variable). Let f be a measurable function on the state space X. Then $f(x_\tau)$ is a random variable on Ω. The integral of this function w.r.t. the measure P_x (if it is meaningful) is the value of the shifted function at the point x. This is expressed by the formula $P_\tau f(x) = E_x f(x_\tau)$, where E_x represents the expectation w.r.t. to P_x. In the case when $\tau = t$ does not depend on ω, the corresponding shift operator is expressed by means of the transition function in the following way: $P_t f(x) = E_x f(x_t) = \int f(y) p_t(x, dy), t \geq 0$. It follows from the Markov principle that $P_t P_s = P_{t+s}$ $(t, s \geq 0)$, i.e. the operators P_t form a semigroup. The right-hand derivative of P_t for $t = 0$ is called the *infinitesimal operator* (or generator) of the process. The *infinitesimal generator* of $\mathcal{P} = (P_t)$ is the possibly unbounded linear operator $\mathcal{A}$ defined by:

$$\mathcal{L}f = \lim_{t \searrow 0} \frac{P_t f - f}{t} \tag{1}$$

The domain $D(\mathcal{L})$ is the subspace of $\mathbf{B}(X)$ for which this limit exists. Under very broad assumptions, the infinitesimal operator uniquely determines the transition function of the process.

Shift Operator For each $t \geq 0$ there exists a map $\theta_t : \Omega \rightarrow \Omega$ called *shift operator* or simply shift such that

$$x_s \circ \theta_t = x_{s+t}, \, \forall s \geq 0. \tag{2}$$

Does a shift exist such that (2) is true? If Ω is the space of all functions on $[0, \infty)$ to X that are trajectories of the process M, we may set $\theta_t(\omega) = x_{t+\cdot}(\omega)$. For an arbitrary Ω, a shift need not exist but it is always possible to construct a shift by enlarging Ω without affecting the probability structure. We do not detail this but rather postulate the existence of a shift as part of our basic machinery for Markov processes.

C. Stochastic Hybrid Systems

We adopt the General Stochastic Hybrid System model presented in [7], [10]. This subsection describes the model and establishes the notation.

Let Q be a set of discrete states. For each $q \in Q$, we consider the Euclidean space $\mathbb{R}^{d(q)}$ with dimension $d(q)$ and

---

[1]A Polish space is a topological space, which is a homeomorphic image of complete separable metric space. The continuous image of a Polish space is called an analytic space.

[2]This is an acronym for the French phrase "continue à droite avec limites à gauche" meaning "continuous on the right with left limits".

we define an *invariant* as an open subset X^q of $\mathbb{R}^{d(q)}$. The hybrid state space is the set $X(Q, d, \mathcal{X}) = \bigcup_{i \in Q} \{i\} \times X^i$ and $x = (i, z^i) \in X(Q, d, \mathcal{X})$ is the hybrid state. The closure of the hybrid state space will be $\overline{X} = X \cup \partial X$, where $\partial X = \bigcup_{i \in Q} \{i\} \times \partial X^i$. It is known that X can be endowed with a metric ρ whose restriction to any component X^i is equivalent to the usual component metric [11]. Then $(X, \mathcal{B}(X))$ is a Borel space (homeomorphic to a Borel subset of a complete separable metric space), where $\mathcal{B}(X)$ is the Borel σ-algebra of X. Let $\mathbf{B}(X)$ be the Banach space of bounded positive measurable functions on X with the norm given by the supremum.

A (General) Stochastic Hybrid System (SHS) is a collection $H = ((Q, d, \mathcal{X}), (b, \sigma), Init, (\lambda, R))$ where
• $(Q, d, \mathcal{X})$ describes the hybrid state space: Q is a countable/finite set of discrete states (modes); $d : Q \to \mathbb{N}$ is a map giving the dimensions of the continuous state spaces; $\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ maps each $q \in Q$ into an open subset X^q of $\mathbb{R}^{d(q)}$;
• (b, σ) provides the coefficients of the diffusion part (continuous dynamics in modes): $b : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot)}$ is a vector field; $\sigma : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot) \times m}$ is a $X^{(\cdot)}$-valued matrix, $m \in \mathbb{N}$,
• $Init$ is the initial probability measure defined on $(X, \mathcal{B}(X))$;
• (λ, R) gives the jumping mechanism: $\lambda : \overline{X}(Q, d, \mathcal{X}) \to \mathbb{R}^+$ is a transition rate function; $R : \overline{X} \times \mathcal{B}(\overline{X}) \to [0, 1]$ is a stochastic kernel that provides the post-jump location.

The realization of an SHS is built as a *Markov string* [7] obtained by the concatenation of the paths of some diffusion processes (z_t^i), $i \in Q$ together with a jumping mechanism given by a family of stopping times (S^i). Let ω_i be a diffusion trajectory, which starts in $(i, z^i) \in X$. Let $t_*(\omega_i)$ be the first hitting time of ∂X^i of the process (x_t^i). Define the function $F(t, \omega_i) = I_{(t < t_*(\omega_i))} \exp(-\int_0^t \lambda(i, z_s^i(\omega_i))ds)$. This function will be the *survivor function* for the stopping time S^i associated to the diffusions (z_t^i).

A stochastic process $x_t = (q(t), z(t))$ is called an *SHS realization* if there exists a sequence of stopping times $T_0 = 0 < T_1 < T_2 \leq \dots$ such that for each $k \in \mathbb{N}$,
• $x_0 = (q_0, z_0^{q_0})$ is a $Q \times X$-valued random variable chosen according to the probability measure $Init$;
• For $t \in [T_k, T_{k+1})$, $q_t = q_{T_k}$ is constant and $z(t)$ is a solution of the stochastic differential equation (SDE):

$$dz(t) = b(q_{T_k}, z(t))dt + \sigma(q_{T_k}, z(t))dW_t \quad (3)$$

where W_t is a the m-dimensional standard Wiener process;
• $T_{k+1} = T_k + S^{i_k}$ where S^{i_k} is chosen according to the survivor function F.
• The post jump location $x(T_{k+1})$ is sampled according to the probability distribution $R\left((q_{T_k}, z(T_{k+1}^-)), \cdot\right)$.

The realization of any SHS, H, under standard assumptions (about the diffusion coefficients, non-Zeno executions, transition measure, etc, see [7] for a detailed presentation) is a strong Markov process. Let $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$ be the strong Markov process associated to H. The sample paths of M are right continuous with left limit, i.e. cadlags.

III. STOCHASTIC REACHABILITY

Let us consider $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$ being a (strong right) Markov process, the realization of a stochastic hybrid system. For this strong Markov process we address a verification problem consisting of the following *stochastic reachability problem*.

Given a target set, the objective of the reachability problem is to compute the probability that the system trajectories from an arbitrary initial state will reach the target set.

Formally, given a set $A \in \mathcal{B}(X)$ and a time horizon $T > 0$, let us define:
$Reach_T(A) = \{\omega \in \Omega \mid \exists t \in [0, T] : x_t(\omega) \in A\}$
$Reach_\infty(A) = \{\omega \in \Omega \mid \exists t \geq 0 : x_t(\omega) \in A\}$.
These two sets are the sets of trajectories of M, which reach the set A (the flow that enters A) in the interval of time $[0, T]$ or $[0, \infty)$. The reachability problem consists of determining the probabilities of such sets. The probabilities of reach events are

$$P(T_A < T) \text{ or } P(T_A < \zeta) \quad (4)$$

where ζ is the life time of M and T_A is the first hitting time of A

$$T_A = \inf\{t > 0 | x_t \in A\} \quad (5)$$

and P is a probability on the measurable space $(\Omega, \mathcal{F})$ of the elementary events associated to M. P can be chosen to be P_x (if we want to consider the trajectories that start in x). Denote by P_A the *hitting operator* associated to the underlying Markov process (x_t), i.e.

$$P_A v = E_x \{v \circ x_{T_A} | T_A < \zeta\} \quad (6)$$

and T_A is given by (5).

Proposition 1: [6] For any $x \in X$ and Borel set $A \in \mathcal{B}(X)$, we have $P_x[Reach_\infty(A)] = P_A 1(x) = P_x[T_A < \zeta]$.

In the context of stochastic reachability we may give a classification of the performance measures for stochastic hybrid systems that can be defined: 1. *Reachability*: The system can reach a certain set of states with a given probability. 2. *Invariance*: The system does not leave a certain set of states with a given probability (viability problem). 3. *Time bounded reachability*: The system can reach a certain set of states within a certain time deadline (horizon time) and probability threshold. 4. *Bounded response*: The system inevitably reaches a certain set of states within a certain time deadline with a given probability.

IV. SYMMETRY REDUCTION: DISCRETE SETTING

In this section, we present briefly the mathematical apparatus of symmetry reduction for discrete probabilistic models as it was developed in the literature [16], [12].

A. Deterministic case

Let $M = (X, \mathcal{R})$ be a transition system with X a finite/countable set of states and a transition relation $\mathcal{R} \subseteq X \times X$. A bijective map (permutation) $\pi : X \to X$ is called an *automorphism* when it preserves the transition relation $\mathcal{R}$, i.e. $(x, x') \in \mathcal{R} \Rightarrow (\pi(x), \pi(x')) \in \mathcal{R}$. Suppose we have

given a group G of such automorphisms under composition of functions. This generates an equivalence relation ϵ on the space X, defined by $(x, x') \in \epsilon$ if there is permutation in G mapping x to x', i.e. if x and x' are symmetric. ϵ is called the *orbit relation*, and its equivalence classes are called *orbits*. Let $\overline{X}$ be the set containing a unique representative state for each equivalence class, we can define a function $rep : X \to \overline{X}$ that selects the corresponding unique representative $rep(x) \in \overline{X}$ for each state $x \in X$ and uses this to define a new transition relation $\overline{\mathcal{R}} = \{(rep(x), rep(x'))|(x, x') \in \mathcal{R}\}$. Since all permutations in G preserve the transition relation $\mathcal{R}$, the quotient transition system $(\overline{X}, \overline{\mathcal{R}})$ is *bisimilar* to the original transition system $(X, \mathcal{R})$.

B. Probabilistic Case

For DTMC, CTMC, the concept of symmetry can be formulated in an analogous way to the non-probabilistic case. Consider permutations of the state space $\pi : X \to X$ that preserve the transition function. For DTMC, we require that

$$P(\pi(x), \pi(x')) = P(x, x'), \forall x, x' \in X. \quad (7)$$

Similarly, for CTMC, we need

$$R(\pi(x), \pi(x')) = R(x, x'), \forall x, x' \in X. \quad (8)$$

As before, we consider a group G of such permutations on X and the corresponding orbit relation ϵ. Using the equivalence w.r.t. ϵ, we define a *reduced state space* $\overline{X}$ containing a unique representative for each orbit and a function $rep : X \to \overline{X}$ which computes the representative for each state. The construction of the quotient model can be done as follows. For a DTMC (X, P) we build the quotient DTMC $(\overline{X}, \overline{P})$, where for each pair of states $\overline{x}, \overline{x}' \in \overline{X}$: $\overline{P}(\overline{x}, \overline{x}') = \sum_{\{x' \in X | rep(x') = \overline{x}'\}} P(\overline{x}, x')$. For a CTMC (X, R), the quotient model is $(\overline{X}, \overline{R})$, where for $\overline{x}, \overline{x}' \in \overline{X}$: $\overline{R}(\overline{x}, \overline{x}') = \sum_{\{x' \in X | rep(x') = \overline{x}'\}} R(\overline{x}, x')$.

In the case of DTMCs and CTMCs, the automorphisms used in symmetry reduction of the state space are *invariance automophisms* since they preserve the transition probabilities (relations (7) and (8)). Applying such automorphims to a chain, the new chain has the same law as the initial one.

V. SYMMETRY REDUCTION VIA THE INVARIANCE GROUP: CONTINUOUS SETTING

Note that in [16], the automorphisms are permutations of the state space, which preserve the transition system relation. For the Markov chains, the automorphisms defined in [16] preserve the probability transition function. For the case of continuous-time continuous space Markov processes, a transition system structure is no longer available (the concept of next state is available only for Markov chains). Therefore, it should be the case that the definition of the concept of *invariance automorphism* to be different: An invariance automorphism must preserve the probabilistic dynamics of the system. Formally, consider a Markov process as a family $\{x_t^x | x \in X\}$ of processes, where x_t^x is the process starting at x. If $\pi : X \to X$ is a homeomorphism, then $\pi(x_t)$ is also a Markov process. The transformation π is called invariance

automorphism of x_t if the process $\pi(x_t^x)$ is identical in law with $x_t^{\pi(x)}$.

A. Invariance

Consider a continuous Markov process defined as in Subsection II-B. Suppose that π is a measurable one to one transformation of the state space $(X, \mathcal{B})$. Then we can identify the Wiener probabilities $\widetilde{P}_x = P_{\pi^{-1}(x)}$ on $\mathcal{F}$. The transformed process is of the form $\widetilde{M} = (\pi x_t, \zeta, \mathcal{F}_t, P_{\pi^{-1}(x)})$. The corresponding transition function is defined by the formula $\widetilde{p}_t(x, \Gamma) = p_t(\pi^{-1}(x), \pi^{-1}\Gamma)$. We say that a Markov process M is *invariant* w.r.t. a transformation π, if the following conditions are satisfied:

• For each $\omega \in \Omega$, there exists $\omega' \in \Omega$ such that

$$\pi x_t(\omega) = x_t(\omega') \text{ for all } 0 \le t < \zeta(\omega) = \zeta(\omega'). \quad (9)$$

• For all $t > 0$, $x \in X$, $\Gamma \in \mathcal{B}$

$$p_t(x, \Gamma) = p_t(\pi^{-1}(x), \pi^{-1}\Gamma). \quad (10)$$

If a Markov process M is invariant w.r.t. π, then the transformed process $\widetilde{M}$ is equivalent to M [13].

If B is a set of trajectories, we can define the shift $\theta_\pi B$ (w.r.t. π) as follows. Put $\omega \in \theta_\pi B$, if ω' can be found such that (9) holds. Then $\theta_\pi \{x_t \in \Gamma\} = \{\pi x_t \in \Gamma\} = \{x_t \in \pi^{-1}\Gamma\}$ for any $t \ge 0$, $\Gamma \subseteq X$.

Theorem 2 (Invariance of the Wiener Probabilities): [13] Let M be a Markov process on the state space $(X, \mathcal{B})$ invariant w.r.t. a transformation π. Then $P_{\pi^{-1}x}(\theta_\pi A) = P_x(A)$, for each $A \in \mathcal{F}$ and $x \in X$.

The transformation π that appears in the definition of invariance can be called *invariance automorphism of M*. An automorphism preserves the transition probabilities and transforms a trajectory of M into another one.

B. Symmetry reduction

Let M be a Markov process on the state space $(X, \mathcal{B})$ and let $\mathcal{G}(X)$ be a group of invariance automorphims of M. Suppose that the group $\mathcal{G}$ preserves the measurable sets. This group generates an equivalence relation ϵ on the space X, defined by $(x, x') \in \epsilon$ if there exists an automorphism in $\mathcal{G}$ mapping x to x'. The subsets $\{\mathcal{G}x\}_{x \in X}$ are called orbits of the group $\mathcal{G}$. Denote by $\widetilde{X} := X/_\mathcal{G}$ the set of all orbits of the group $\mathcal{G}$. Denote by γ the projection map from X to $\widetilde{X}$ defined by $\gamma x := \{\mathcal{G}x\}$. Let $\widetilde{\mathcal{B}} := \gamma \mathcal{B}$. then γ is a measurable transformation of $(X, \mathcal{B})$ into $(\widetilde{X}, \widetilde{\mathcal{B}})$. The invariance of M w.r.t. to the automorphisms of $\mathcal{G}$ enables us to construct a Markov process on the state space $(\widetilde{X}, \widetilde{\mathcal{B}})$ from the Markov process M, using the transformation γ [13]. Denote by $M/_\mathcal{G}$ this new Markov process. $M/_\mathcal{G}$ is obtained from M by *symmetry reduction of the state space w.r.t. the group $\mathcal{G}$*.

We can define a *reduced state space* or a *fundamental domain* for the group $\mathcal{G}$ as follows. A set $\overline{X} \subset X$ is a reduced state space for $\mathcal{G}$ if one and only one point belonging to $\overline{X}$ can be found in each orbit $\{\mathcal{G}x\}$. Then associating the class $\{\mathcal{G}x\}$ with this point we obtain a one to one mapping $\beta : \widetilde{X} \to \overline{X}$. Naturally, we can then define $rep : X \to \overline{X}$;

$rep := \beta \circ \gamma$. Assume that $\overline{X} \in \mathcal{B}$ and set $\overline{\mathcal{B}} := \mathcal{B}(\overline{X})$. Then $\beta\widetilde{\mathcal{B}} = \overline{\mathcal{B}}$ and $\beta^{-1}\overline{\mathcal{B}} = \widetilde{\mathcal{B}}$. This says that it is possible to identify the space $(\widetilde{X}, \widetilde{\mathcal{B}})$ with the space $(\overline{X}, \overline{\mathcal{B}})$ and consider the process $M/_{\mathcal{G}}$ to be given on $(\overline{X}, \overline{\mathcal{B}})$. The Markov process M under P_x is equivalent with the Markov process $M/_{\mathcal{G}}$ under $\overline{P}_{rep(x)}$.

Example 1: [13] Suppose that the state space is $\mathbb{R}^n$ with the Borel σ-algebra. Let $\mathcal{G}$ be the group of all orthogonal transformation of the state space. Select an arbitrary unit vector $e \in \mathbb{R}^n$. Then the semigroup αe $(\alpha \geq 0)$ is the reduced state space for the group $\mathcal{G}$. Therefore, to each Markov process M on $\mathbb{R}^n$ invariant w.r.t. the group $\mathcal{G}$, there corresponds a process $M/_{\mathcal{G}}$ on the half-line.

VI. SYMMETRY REDUCTION VIA SYMMETRY GROUPS: CONTINUOUS SETTING

When we are considering complex Markov processes as those that appear as semantics of SHS, the symmetry reduction described in the Section V might be difficult to apply. We need to find an appropriate transformation group $\mathcal{G}$ whose elements are also automorphisms for the diffusion components. As well, we need to check properties like the invariance of the transition rate λ or of the stochastic kernel R (that appear in the definition of SHS) w.r.t. the elements of $\mathcal{G}$. This might be a difficult task taking in consideration the structure of the SHS executions (trajectories). In order to have two "symmetric" trajectories, we need some symmetry also for their diffusion parts. But if we start in a mode with two symmetric diffusion paths, after the first jump we may get some asymmetric paths in another mode or in two different modes.

Our novelty is to consider transformation groups for which we have the symmetry properties of some observation functions instead of invariance groups. These groups are *symmetry groups* and their elements are some particular *symmetry automorphims*. Formally, consider a Markov process $\{x_t^x | x \in X\}$. A homeomorphism $\pi : X \to X$ is called symmetry automorphism of x_t if the process $\pi(x_t^x)$ is identical in law with $x_t^{\pi(x)}$ after a time change.

The line of this section can be described as follows. We present first the concept of time change for Markov processes. Then we define formally the observation functions as expectations of some random variables over the paths (that provide "observations" about the trajectories). The next step is to define the observation automorphisms as permutations of the state space that preserve the observation functions. The group of such automorphisms is used thereafter to "reduce the state space" considering the quotient space w.r.t. the equivalence relation induced by this group. At the end, we show that this symmetry reduction of state space preserves the reach set probabilities.

A. Time change

Let us recall briefly the definition of time changes for Markov processes [20]. A real valued process A_t is called an *additive functional* of (x_t) if it is adapted to the natural filtration of (x_t) and satisfies $A_0 = 0$ and $A_{t+s} = A_t + A_s \circ \theta_t$, where θ_t

is the shift operator defined by (2). Suppose that an additive functional has continuous strictly continuous paths. Let τ_t be the inverse of A_t considered as a function of t. τ_t is called a time-change process of (x_t). The process (x_{τ_t}) (which is also a Markov process) has the same physical paths as (x_t), but runs according to a different clock.

Let $a(x)$ a positive continuous function on X bounded away from 0. Then $A_t = \int_0^t a(x_s)ds$ is an additive functional and a is called the density of A_t. If τ_t is the inverse of A_t, then the time-changed process (x_{τ_t}) is said to be obtained from (x_t) by the time change with density a. In this case, the generator of the time-changed process is given by (see [20], p.278): $\widetilde{\mathcal{L}}f(x) = a(x)^{-1}\mathcal{L}f(x); f \in D(\mathcal{L})$. Let us exemplify the time change with density a for a finite Markov chain. Denote by Q the associated generator (stochastic) matrix (each element q_{ij} represents the transition rate from i to j). The matrix $\widetilde{Q}$ corresponding to the time change Markov chain is $\widetilde{q}_{ij} = a(i)^{-1}q_{ij}$. In terms of jump-hold description of the chain, the time change can be specify as follows: when the time-changed chain visits i, it resides there for an exponential amount of time with mean $a(i)/q(i)$ compared with the mean $1/q(i)$ for the original chain.

Two processes that differ by a time change have the same hitting distribution, by the Blumenthal-Getoor-McKean Theorem (Ch. 5 of [5], [15]). Then, according to the Prop. 1, two such processes have the same reach set probabilities, so they are "bisimilar".

B. Observability over the paths

We suppose that the trajectories $x : [0, \infty) \to X$ of M are right continuous and have left limits. We consider $\Omega = D_X[0, \infty)$, the set of all these paths (i.e. the space of all *cadlag functions* from $[0, \infty)$ to X). Such functions are known also as *Skorokhod functions*. A topological structure (topology) on the space $D_X[0, \infty)$ has been introduced by Skorokhod as an alternative to the topology of uniform convergence [14].

In the following, we define a special class of functions called *observation functions* for the Markov process M. These functions play the role of some logic formulas over the trajectories. First we define the observation random variables. Taking the expectations of such random variables represents a technique to generate observation functions. This technique provides also intuitions about the meaning of these functions.

A nonnegative function $\eta : \Omega \to \mathbb{R}_+$ is said to be an *observation random variable* for the process M, if: (i) the function η is measurable; (ii) the value of η on the shifted trajectory is less than the value of η on the whole trajectory, i.e. $\eta(\theta_t\omega) \leq \eta(\omega)$ for all $0 \leq t < \zeta(\omega)$; (iii) the function $\eta(\theta_t\omega)$ is *right-continuous* in $t \in [0, \zeta(\omega))$ for all ω.

In the language of [13], the observation random variables are called excessive random variables. Some well known observation random variables are recorded as follows:

● Entrance/hitting time: For any measurable set $A \subset X_\partial$, one can define *the first entrance time into* A: $D_A(\omega) = \inf\{t \geq 0 | x_t(\omega) \in A\}$, and T_A *the first hitting time of* A given by (5). They are related by: $T_A = \lim \downarrow (s + D_A \circ \theta_s)$.

• Exit Time: For any measurable set $A \subset X$, one can define *the first exit time from A* as the first hitting time of its complement $(X \backslash A)$.

• Last Exit Time: For any measurable set $A \subset X$, one can define *the last exit time of A* or "quitting time of A" as follows: $L_A(\omega) = \sup\{t \geq 0 | x_t(\omega) \notin A\}$. The last exit time can be used to characterize concepts like transience and recurrence for the measurable sets.

• Sojourn Time: For any $A \in \mathcal{B}(X)$, the *sojourn time* on A is given by $S_A(\omega) = \int_0^{L_A(\omega)} I_A(x_t(\omega))dt$. The sojourn time of a set can be employed to define the occupation measure of that set.

Let η be an excessive random variable, satisfying the additional requirement: $0 < E_x \eta < \infty$, for all $x \in X$.

Proposition 3: Let M be a strong Markov process. If η is an observation random variable, then $f(x) = E_x \eta$ satisfies the following conditions: (a) $E_x f(x_\tau) \leq f(x)$, for all $x \in X$ and for any stopping time τ; (b) $\lim_{n \to \infty} E_x f(x_{\tau_n}) = f(x)$, for any $x \in X$ and any sequence of stopping times τ_n such that $P_x(\tau_n \searrow 0) = 1$.

Example 2: If h is an arbitrary non-negative $\mathcal{B}$-measurable function then $\eta = \int_0^\zeta h(x_t)dt$, where ζ is the life time of M, is an observation random variable.

The set of non-negative measurable functions f that satisfy the conditions (a) and (b) from the Prop. 3 may be larger than the set of such functions provided by observation random variables. For instance, these properties remain true for limits of such function

Definition 1 (Observation Function): A non-negative measurable function $f : X \to [0, \infty]$ is called observation function for the process M if the conditions (a) and (b) from the Prop. 3 are fulfilled.

Theorem 4: [13] A non-negative measurable function $f : X \to [0, \infty]$ is an observation function for a strong Markov process M if and only if the following conditions w.r.t. the operator semigroup $\mathcal{P}$ are satisfied:

(i) $P_t f(x) \leq f(x)$ for all $t \geq 0$, $x \in X$; (ii) $P_t f(x) \to f(x)$ as $t \searrow 0$, for every $x \in X$.

Th.4 shows that our observation functions are exactly 0-excessive functions defined in the context of Markov processes. Let us denote by $\mathcal{O}b(M)$ the set of observation (or 0-excessive) functions associated to M.

Recall that a function f is called $\alpha-$*excessive* $(\alpha \geq 0)$ w.r.t. the semigroup (P_t) if it is measurable, non-negative and $e^{-\alpha t} P_t f \leq f$, for all $t \geq 0$ and $e^{-\alpha t} P_t f \nearrow f$ as $t \searrow 0$. Let $\mathcal{E}_M^\alpha$ be the set of all excessive functions associated to M. According to the Blumenthal-Getoor-McKean theorem [5], the cone of excessive functions determines the process up to a time change. For a better understanding of the concept of 0-excessive function (observation function) we instantiate M with a CTMC defined on a countable state space I whose generator is denoted by Q. A sequence $C = \{C(i)\}$ of nonnegative finite numbers indexed by I is called a $P(t)$-excessive if $P(t)C \leq C$, for all t. The following characterization of the excessive functions associated to a CTMC is a classical result [20].

Proposition 5: C is $P(t)$-excessive if and only if $C \geq 0$ and $QC \leq 0$.

Remark 1: We assume also that M is *transient* [3]. This means that there exists a strictly positive Borel measurable function q such that $Vq := \int_0^\infty P_t q(x)dt$ is bounded. The transience hypothesis guarantees that the cone $\mathcal{O}b(M)$ is rich enough to be used. The importance of the concept of transience for Markov chains is pointed out in [3].

C. Symmetry Group

Let us consider a transient Markov process M with the state space $(X, \mathcal{B})$ (M is thought of as the realization of an SHS, H). Let $\mathcal{S}(X)$ be the group of all homeomorphisms $\varphi : X \to X$, i.e. all bijective maps φ such that φ, φ^{-1} are $\mathcal{B}(X)$-measurable. When X is finite, $\mathcal{S}(X)$ is the set of (finite) symmetries of X.

Any symmetry[4] of X induces a symmetry of the group of bounded measurable functions as follows. Let $* : \mathcal{S}(X) \to Perm[\mathbf{B}(X)]$ be the *action* $\mathcal{S}(X)$ to $\mathbf{B}(X)$ defined by $*(\varphi) = \varphi^* : \mathbf{B}(X) \to \mathbf{B}(X)$, where φ^* is the linear operator on $\mathbf{B}(X)$ given by

$$\varphi^* f = f \circ \varphi. \tag{11}$$

The range of $*$ is enclosed in $Perm[\mathbf{B}(X)]$ (the symmetry group of $\mathbf{B}(X)$). This fact is justified by the invertibility of φ^*. The invertibility of φ^* can be derived from the bijectivity of $\varphi \in \mathcal{S}(X)$ (since we have $(\varphi^*)^{-1} = (\varphi^{-1})^*$). Therefore, φ^* can be thought of as a symmetry of $\mathbf{B}(X)$ for each φ given in the appropriate set.

The observation functions are clearly Borel measurable functions (i.e. $\mathcal{O}b(M) \subset \mathbf{B}(X)$). We can not define the action of $\mathcal{S}(X)$ to $\mathcal{O}b(M)$ using formula (11) because the result of composition in (11) is not always an excessive function.

Therefore it is necessary to consider those subgroups of the state space symmetries such that we can define the action of these subgroups on the semigroup of the observation functions $\mathcal{O}b(M)$.

Consider the *maximal subgroup of symmetries* of the state space X, denoted by $\mathcal{H}$, such that the *action of $\mathcal{H}$ to $\mathcal{O}b(M)$* denoted also by '$*$' can be defined: $* : \mathcal{H} \to Perm[\mathcal{O}b(M)]$ as the appropriate restriction of (11). The elements of $\mathcal{H}$ 'preserve' through '$*$' the observation functions. In other words, we have the invariance of the observations w.r.t. the elements of $\mathcal{H}$. These observations could be interpreted as well as some stochastic specifications of the system. $\mathcal{H}$ is not necessary to be taken as the maximal subgroup of symmetries with this property. Naturally, the elements of $\mathcal{H}$ will be called *observation automorphisms* of M.

In particular, using the Proposition 5, it is easy to prove that the automorphisms defined for Markov chains in [16] preserve, as well, the excessive functions, i.e. are observation automorphisms.

---

[3]The transience of M means that any process trajectory which will visit a Borel measurable set of the state space, it will leave it after a finite time.

[4]Here, permutation is used with the sense of one-to-one correspondence or bijection.

Using $\mathcal{H}$, an equivalence relation $\mathcal{O} \subset X \times X$, called *observation relation*, can be defined on the state space X as follows.

Definition 2: Two states x, y are in the same orbit, written $x\mathcal{O}y$, if and only if there exists an observation automorphism $\varphi \in \mathcal{H}$ such that $\varphi(x) = y$.

Let us denote by $[x]$ the equivalence class containing the point x in X. The equivalent classes of $\mathcal{O}$ are called *orbits*. It is clear that an orbit $[x]$ can be described as $[x] = \{\varphi(x)|\varphi \in \mathcal{H}\} = \{\mathcal{H}x\}$. Let $X/_\mathcal{O}$ denote the set of orbits, and let $\Pi_\mathcal{O}$ the canonical projection $\Pi_\mathcal{O} : X \to X/_\mathcal{O}$, $\Pi_\mathcal{O}(x) = [x]$. The space $X/_\mathcal{O}$ will be equipped with the quotient topology by declaring a set $A \subset X/_\mathcal{O}$ to be open if and only if $\Pi_\mathcal{O}^{-1}(A)$ is open in X. $\Pi_\mathcal{O}$ is a continuous map w.r.t. the initial topology of X and the quotient topology of $X/_\mathcal{O}$.

D. Symmetry Reduction

In this subsection, we show that the observation automorphisms are, in fact, symmetry automorphims, so they preserve the hitting distributions. The consequence of this fact is that the reach set probabilities (4) are preserved through the observation automorphisms. Moreover, since the reach set probabilities are preserved, the observation relation $\mathcal{O}$ is nothing else, but a *bisimulation relation* on the state space.

Proposition 6: Let $g : X/_\mathcal{O} \to \mathbb{R}$ be a $\mathcal{B}(X/_\mathcal{O})$-measurable and let $E = \Pi_\mathcal{O}^{-1}(A)$ for some $A \in \mathcal{B}(X/_\mathcal{O})$. Then the following equality holds

$$P_E = \varphi^* \circ P_A, \forall \varphi \in \mathcal{H} \tag{12}$$

applied to all functions $f : X \to \mathbb{R}$, $f = g \circ \Pi_\mathcal{O}$.

To prove this proposition we need the following lemma.

Lemma 7: If $f \in \mathcal{O}b(M)$ and $\varphi \in \mathcal{H}$ then

$$P_E f = \varphi^*[P_F(\vartheta)] \tag{13}$$

where $F = \varphi(E)$; $\vartheta = \varphi^{-1*}f$, the action of '$*$' is given by (11) and P_F is the hitting operator associated to F.

Proof of Prop. 6. If in Lemma 7, we let $f = g \circ \Pi_\mathcal{O}$, then $\vartheta = \varphi^{*-1}f = f \circ \varphi^{-1} = g \circ \Pi_\mathcal{O} \circ \varphi^{-1} = f$. More, $\varphi(\Pi_\mathcal{O}^{-1}(A)) = \Pi_\mathcal{O}^{-1}(A)$, so the proposition follows from the above lemma. ∎

Corollary 8: Any observation automorphism $\varphi \in \mathcal{H}$ for M is a symmetry automorphism, i.e. M and $\varphi(M)$ differ by a time change, then they have the same hitting distributions.

Formula (12) shows that the function $P_E f$ (where $f = g \circ \Pi_\mathcal{O}$) is constant on the equivalent classes w.r.t. $\mathcal{O}$. Then it makes sense to define a collection of operators (Q_A) on $(X/_\mathcal{O}, \mathcal{B}(X/_\mathcal{O}))$ by setting

$$Q_A g([x]) = P_E(g \circ \Pi_\mathcal{O})(x) \tag{14}$$

where $E = \Pi_\mathcal{O}^{-1}(A)$. Proposition 6 allows to use any representative x of $[x]$ in the right side of (14). It is easy to check that $Q_A Q_B = Q_B$ if A and B are open sets of $X/_\mathcal{O}$ with $B \subset A$. Under some supplementary hypotheses one can construct a Markov process $M/_\mathcal{O} = ([x]_t, Q_{[x]})$ with these hitting operators [5]. $M/_\mathcal{O}$ is obtained from M by *symmetry reduction of the state space w.r.t. the group $\mathcal{H}$ and the set of observations $\mathcal{O}b(M)$.*

E. Stochastic Bisimulation

For a continuous time continuous space Markov process M with the state space X, an equivalence relation $\mathcal{R}$ on X is called *(strong) bisimulation* if for $x\mathcal{R}y$ we have $p_t(x, A) = p_t(y, A), \forall t > 0, \forall A \in \mathcal{B}(X/_R)$, where $p_t(x, A)$, $x \in X$ are the transition probabilities of M and $\mathcal{B}(X/_R)$ represent the σ-algebra of measurable sets closed w.r.t. $\mathcal{R}$ (see [8] for the categorical version of this bisimulation concept). This variant of strong bisimulation considers two states to be equivalent if their 'cumulative' probability to 'jump' to any set of equivalent classes (that this relation induces) is the same. This is hard to be checked in practice since the time t runs continuously. Therefore, to construct a robust bisimulation relation on X it is necessary to use other parametrizations of M, that preserves only the measures of interest for the Markov process M.

In the following we briefly present the concept of bisimulation defined in [9]. This concept is more robust and it can be characterized by an interesting pseudometric [9].

Suppose we have given a Markov process M on the state space X, w.r.t. a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Assume that $\mathcal{R} \subset X \times X$ is an equivalence relation such that the quotient process $M|_\mathcal{R}$ is still a Markov process with the state space $X/_\mathcal{R}$, w.r.t. a probability space $(\Omega, \mathcal{F}, \mathbf{Q})$. That means that the projection map $\Pi_\mathcal{R}$ associated to $\mathcal{R}$ is a Markov function [9]. A relation $\mathcal{R}$ is called *(observational) bisimulation* on X if for any $A \in \mathcal{B}(X/_\mathcal{R})$ we have that $\mathbf{P}[T_E < \infty] = \mathbf{Q}[T_A < \infty]$, where $E = \Pi_\mathcal{R}^{-1}(A)$ (i.e. the reach set probabilities of the process M and $M|_\mathcal{R}$ are equal). Note that when we consider discrete probabilistic models (like DTMC, CTMC) the bisimulation concept becomes the bisimulation considered in [2].

Theorem 9: The observation relation $\mathcal{O}$ is a bisimulation relation on $(X, \mathcal{B})$ for the Markov process M.

Th. 9 is a simple consequence of the Prop. 6, but its statement is very important in the context of stochastic reachability. It states that the symmetry reduction of the state space defined via observation automorphisms represents a sound approach that can be used further in stochastic model checking.

VII. TOWARDS SYMMETRY REDUCTION FOR SHS

In this section, we discuss how the symmetry reduction techniques described in Sections V and VI can be further adapted in the framework of stochastic hybrid systems. We have already pointed out that the fact that symmetry reduction via invariance groups is not a realistic choice for SHS due to the jumping mechanism between the discrete locations. One way to deal with this method is to apply symmetry reduction locally in each mode for the corresponding diffusion process and then to find the appropriate composition mechanism for these local abstractions, in order to obtain the global abstraction of the given SHS.

The second symmetry reduction technique (via a group of observation automorphisms, Section VI) might be a valuable method to reduce the state space of a stochastic hybrid system. The efficiency of this method depends pretty much

on our ability to choose the generators of the semigroup of observation functions. Considering the connection between the semigroup of operators and the infinitesimal generator of a Markov process (Hille-Yosida theorem [14]), based on the Th.4, one can easily obtain characterizations of the observation functions in terms of the generator (see also [5] for different characterizations of the excessive elements).

The infinitesimal generator of the realization of an SHS H is an integro-differential operator. In [10], it was proved that the extended generator of an SHS has the following expression:

$$\mathcal{L}f(x) = \mathcal{L}_{cont}f(x) + \mathcal{L}_{dis}f(x) \qquad (15)$$

where $\mathcal{L}_{cont}f(x)$ has the standard form of the diffusion infinitesimal operator and $\mathcal{L}_{dis}f(x) = \lambda(x)\int_{\overline{X}}(f(y) - f(x))R(x, dy)$ (typical generator of a jump process). The domain $D(\mathcal{L})$ contains at least the set of second order differentiable functions that satisfy the following boundary condition: $f(x) = \int_{\overline{X}} f(y)R(x, dy)$, $x \in \partial X$.

For any $\varphi \in \mathcal{S}(X)$ (where $\mathcal{S}(X)$ is defined as in Subsection VI-C), the generator of $\varphi(M)$ is given by $\mathcal{L}^{\varphi}f = \varphi_*[\mathcal{L}(\varphi^*f)]$, where $\varphi_*f := f \circ \varphi^{-1}$. Then we can define the *invariance group* $Inv(\mathcal{L}) := \{\varphi \in \mathcal{S}(X)|\mathcal{L}^{\varphi} = \mathcal{L}\}$. Analogously, the *symmetry group* can be defined taking into account the results from Subsection VI-A as follows: $Sym(\mathcal{L}) := \{\varphi \in \mathcal{S}(X)|\exists \beta \in C_0(X), \beta > 0, \mathcal{L}^{\varphi} = \beta\mathcal{L}\}$. Clearly, $Inv(\mathcal{L}) \subset Sym(\mathcal{L})$. To apply symmetry reduction to SHS, we need the assumption that there is a group of symmetries acting uniformly on the diffusion processes of different discrete modes, and the transition rate λ and the stochastic kernel R are 'invariant' w.r.t. these symmetries. Finding appropriate symmetry automorphisms for SHS might be a difficult and challenging task. In the first step, considering the expression of the SHS generator (15), it is clear that we need to consider symmetry groups for the continuous dynamics of an SHS. Characterizations of the invariance group and symmetry group for diffusion processes can be given using the *isometry group* (that consists of transformations which leave the metric invariant) and the *conformal group* (that consists of transformations which do not change the angles) [18]. In the second step, consider φ a symmetry/invariant automorphism for the diffusion part and observe that $\mathcal{L}_{dis}^{\varphi}f(x) = \lambda(\varphi^{-1}(x))\{\int_{\overline{X}} f(\varphi(y))R(\varphi^{-1}(x), dy) - f(x)\}$.

Proposition 10: φ is an invariant automorphism for the whole process M_H (realization of H) iff $\varphi_*\lambda = \lambda$, $\int_{\overline{X}} f(\varphi(y))R(\varphi^{-1}(x) = \int_{\overline{X}} f(y)R(x, dy)$, $f \in D(\mathcal{L})$.

A similar condition can be written for a symmetry automorphism. In a following paper, we will investigate further these conditions in order to find necessary conditions for a transformation group to be an appropriate subgroup of $Inv(\mathcal{L})$ or $Sym(\mathcal{L})$, where $\mathcal{L}$ is the infinitesimal generator of an SHS.

VIII. CONCLUSIONS

Modelling with SHS is very fashionable in engineering because of the versatile randomisation techniques it offers. However, this paradigm is less popular in computer science due to the inherent complexity of the formal verification of safety properties. In this work, we address the verification issue by investigating how probabilistic model checking techniques from computer science can be extending for SHS. We have mainly presented two techniques for symmetry reduction of the state space for continuous probabilistic systems. Both of them are based on the same methodology to obtain the reduced state space: choose an appropriate group of permutations of the state space (the invariant group and the symmetry group) and then construct the quotient space w.r.t. this group. We have also proved that the reduced quotient model is bisimulation-equivalent to the original model. Finally, both techniques are discussed for stochastic hybrid systems.

REFERENCES

[1] Ames, A.D., Sastry, S.: *Hybrid Geometric Reduction of Hybrid Systems*. In Proc. IEEE CDC (2006).

[2] Baier, C., Katoen, J.-P., Hermanns, H., Wolf, V.: *Comparative Branching-time Semantics for Markov Chains*. Information and Computation **200** (2005): 149-214.

[3] Baier, C., Haverkort, B.R., Hermanns, H., Katoen, J.-P.: *Model Checking Continuous Time Markov Chains by Transient Analysis*. In E.A. Emerson and A.P. Sistla (Eds.). "Computer Aided Verification." Springer LNCS **1855** (2000): 358-372.

[4] Blom, H.A.P., Lygeros, J. (Eds.): "*Stochastic Hybrid Systems: Theory and Safety Critical Applications*". LNCIS **337** (2006).

[5] Blumenthal, R.M., Getoor, R.K.: "*Markov Processes and Potential Theory*", Academic Press, New York and London (1968).

[6] Bujorianu, M.L., Lygeros, J.: *New Insights on Stochastic Reachability*. Proc. 46th Conference in Decision and Control (2007).

[7] Bujorianu, M.L., Lygeros, J.: *Towards Modelling of General Stochastic Hybrid Systems*. In [4]: 3-30.

[8] Bujorianu, M.L., Lygeros, J., Bujorianu, M.C.: *Bisimulation for General Stochastic Hybrid Systems*. In Morari, M., Thiele, L. (Eds.): "*Proc. Hybrid Systems: Computation and Control*" 8th International Workshop, Springer LNCS 3414 (2005): 198-216.

[9] Bujorianu, M.L., Lygeros, J., Bujorianu, M.C.: *Abstractions of Stochastic Hybrid System*. Proc. 44th Conference in Decision and Control. IEEE Press (2005).

[10] Bujorianu, M.L., Lygeros, J.: *General Stochastic Hybrid Systems: Modelling and Optimal Control*. Proc. 43th Conference in Decision and Control (2004).

[11] Davis, M.H.A.: "*Markov Models and Optimization*" Chapman & Hall, (1993).

[12] Donaldson, A.F., Miller, A.: *Symmetry Reduction for Probabilistic Model Checking Using Generic Representatives*. ATVA (2006): 9-23

[13] Dynkin, E.B.: "*Markov Processes. Vol. I&II*". Springer Verlag (1965).

[14] Ethier, S.N., Kurtz, T.G.: "*Markov Processes: Characterization and Convergence*". New York: John Wiley and Sons, (1986).

[15] Fitzsimmons, P.J., Getoor, R. K., Sharpe, M. J.: *The Blumenthal-Getoor-McKean Theorem Revisited*. In: Seminar on Stochastic Processes 1989, Birkhäuser, Boston, (1990): 35-57.

[16] Kwiatkowska, M., Norman, G., Parker, D.: *Symmetry Reduction for Probabilistic Model Checking*. In Proc. 18th International Conference on Computer Aided Verification (CAV'06), LNCS **4144**, (2006): 234-248.

[17] Kwiatkowska, M., Norman, G., Parker, D., Sproston, J.: *Performance Analysis of Probabilistic Timed Automata using Digital Clocks*. Formal Methods in System Design. Springer. (2003): 105-120.

[18] Liao, M.: *Symmetry Groups of Markov Processes*. Ann. Prob. **20** (2), (1992): 563-578.

[19] Marsden, J. E. , Ratiu, T. S. : "*Introduction to Mechanics and Symmetry*". Texts in Applied Mathematics. Springer, (1999), vol. 17.

[20] Williams, D., Rogers, L.C.G.: "*Diffusions, Markov Processes, and Martingales: Volume 1, Foundations*". Cambridge Mathematical Library (2000).