Model Checking HML On Piecewise-Constant Inhomogeneous Markov Chains*

Joost-Pieter Katoen and Alexandru Mereacre

Software Modeling and Verification, RWTH Aachen University, Germany
{katoen,mereacre}@cs.rwth-aachen.de

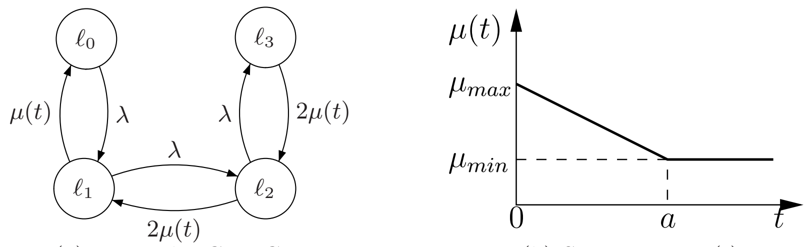
Abstract. This paper presents a stochastic variant of Hennessy-Milner logic that is interpreted over (state-labeled) inhomogeneous continuous-time Markov chains (ICTMCs), i.e., Markov chains in which transition rates are functions over time t. For piecewise constant rate functions, the model-checking problem is shown to be reducible to finding the zeros of an exponential polynomial. Using Sturm sequences and Newton's method, we obtain an approximative model-checking algorithm which is linear in the size of the ICTMC, logarithmic in the number of bits precision, and exponential in the nesting depth of the formula.

1 Introduction

Continuous-time Markov chains (CTMCs) are applied in a large range of applications, ranging from transportation systems to systems biology, and are a popular model in performance and dependability analysis. These Markov chains are typically homogeneous, i.e., the rates that determine the speed of changing state as well as the probabilistic nature of mode transitions are constant. However, in some situations constant rates do not adequately model real behavior. This applies, e.g., to failure rates of hardware components (that usually depend on the component's age), battery depletion (where the power extraction rate non-linearly depends on the remaining amount of energy), and random phenomena that are subject to environmental influences such as temperature. In these circumstances, Markov models with *inhomogeneous* rates, i.e., rates that are time-varying functions, are more appropriate.

Whereas temporal logics and accompanying model-checking algorithms have been developed for CTMCs [5, 4], and have resulted in a number of successful model checkers such as PRISM [16] and MRMC [15], the verification of time-inhomogeneous CTMCs (ICTMCs) has – to the best of our knowledge – not yet been investigated. This paper presents an initial step in that direction by presenting a stochastic variant of the well-known Hennessy-Milner Logic [11] (HML) for ICTMCs. The main ingredient is a simple probabilistic real-time extension of the modal operator $\langle\Phi\rangle$ in (state-based) HML: the formula $\langle\Phi\rangle^{I}_{\geq p}$ asserts that a Φ-state is reachable in the time interval I with likelihood at least p. The main

* This research has taken place as part of the Research Training Group 1298 ALGOSYN funded by the German Research Council.



(a) Example ICTMC (b) Service rate $\mu(t)$

Fig. 1. Queue with three capacities and two servers.

technical difficulty is that the semantics of the stochastic variant of HML has to be defined on the underlying continuous state space of an ICTMC. This is similar to the semantics of timed CTL [2] over timed automata [3] which is typically interpreted over infinite-state timed transition systems. The adequacy of this extension is justified by the fact that, as for the discrete probabilistic variant of HML [18], logical equivalence corresponds to strong bisimulation. Opposed to CTMC model checking (where all rate functions are constant), restrictions have to be imposed on the rate functions in order to enable (approximate) model-checking algorithms for ICTMCs. It is shown that verifying our variant of HML for rate functions that are piecewise constant boils down to determining the zeros of an exponential polynomial. Using Laguerre's theorem [17] it can be established that this polynomial has at most five such zeros. By transforming the exponential polynomial into an equivalent (square-free) ordinary polynomial, Sturm sequences, as well as the well-known Newton's method are applied to obtain these zeros. This results in an approximative verification algorithm for stochastic HML which is exponential in the nesting depth of the formula (i.e., the number of $\langle\Phi\rangle^{I}_{\geq p}$ formulas in sequence), linear in the size of the ICTMC, linear in the number of pieces of a rate function, and logarithmic in the number of bits precision of Newton's method.

2 Preliminaries

Definition 1 (ICTMC). *An* inhomogeneous continuous-time Markov chain *(*ICTMC*) is a structure* $\mathcal{C} = (\mathbb{L}, \ell_0, \mathbf{R}(t), AP, L)$ *such that:* $\mathbb{L}$ *is a finite set of* n *locations,* $\ell_0 \in \mathbb{L}$ *is the initial location,* $\mathbf{R}(t) = [R_{\ell,\ell'}(t)] \in \mathbb{R}^{n\times n}_{\geqslant 0}$ *is a time-dependent rate matrix, where* $R_{\ell,\ell'}(t)$ *is the rate between locations* $\ell, \ell' \in \mathbb{L}$ *at time* $t \in \mathbb{R}_{\geqslant 0}$*,* AP *is a finite set of atomic propositions and* L *is the labeling function defined as* $L : \mathbb{L} \to 2^{AP}$*.*

Let diagonal matrix $\mathbf{E}(t) = \text{diag}\,[E_\ell(t)] \in \mathbb{R}^{n\times n}_{\geqslant 0}$, where $E_\ell(t) = \sum_{\ell'\in\mathbb{L}} R_{\ell,\ell'}(t)$ for all $\ell, \ell' \in \mathbb{L}$ i.e., $E_\ell(t)$ is the total exit rate of location ℓ at time t. We sometimes write $\ell \overset{\lambda(t)}{\to} \ell'$ as shorthand for $R_{\ell,\ell'}(t) = \lambda(t)$. Note that the only requirement for rates and exit rates is that they are integrable. If all rates (and thus exit rates) are constant, we obtain a CTMC.

The state of an ICTMC is determined by the current state of control (i.e., location), and the current instant of time. A state ξ of ICTMC $\mathcal{C}$ is a tuple (ℓ, x) where $\ell \in \mathbb{L}$ indicates the current location and $x \in \mathbb{R}_{\geqslant 0}$ the current time instant. The state space $\mathbb{E}$ of an ICTMC is given by a disjoint union of subsets $\mathbb{E}_\ell$ of $\mathbb{R}_{\geqslant 0}$ for each location ℓ such that:

$$\mathbb{E} = \coprod_{\ell \in \mathbb{L}} \mathbb{E}_\ell = \bigcup_{\ell \in \mathbb{L}} \{(\ell, x) \mid x \in \mathbb{R}_{\geqslant 0}\}.$$

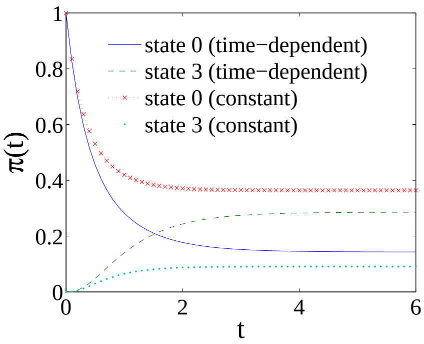


Fig. 2. Transient distribution for ℓ_0 and ℓ_3.

Let $\mathcal{E}$ denote the σ-field of subsets A of $\mathbb{E}$ which take the form $A = \coprod_{\ell \in \mathbb{L}} A_\ell$, where A_ℓ is a Borel set of $\mathbb{E}_\ell$ (defined as above). As the set $\mathbb{R}_{\geqslant 0}$ denotes the set of possible time points, the initial state $\xi_0 \in \mathbb{E}$ of $\mathcal{C}$ becomes $\xi_0 = (\ell_0, 0)$. For any state $\xi = (\ell, x)$ the projection $\xi_{\mathbb{L}}$ yields $\ell \in \mathbb{L}$, and projection $\xi_{\mathbb{R}}$ yields $x \in \mathbb{R}_{\geqslant 0}$. These projection functions are lifted to sets of states in a pointwise manner. For a set of states $A \subseteq \mathbb{E}$ and location $\ell \in \mathbb{L}$, let $A_\ell^{\mathbb{R}}$ denote the set $\{x \in \mathbb{R}_{\geqslant 0} \mid \xi \in A, \xi_{\mathbb{L}} = \ell, \xi_{\mathbb{R}} = x\}$.

Example 1. Fig. 1(a) shows a queue with three capacities and two servers modeled by an ICTMC. The customers arrive as a Poisson process with rate λ and the service rate is a function $\mu(t)$ (see Fig. 1(b)). Initially the service rate starts at μ_{max} and decreases linearly until μ_{min} at time $t = a$. From that moment on, all customers are served with constant rate μ_{min}.

The transition probability (kernel) $\Pr : \mathbb{E} \times \mathcal{E} \to [0, 1]$ of an ICTMC is a probability measure $\Pr(\xi, \cdot)$ on $(\mathbb{E}, \mathcal{E})$ for each fixed $\xi \in \mathbb{E}$, and $\Pr(\cdot, A)$ is a measurable function for each fixed $A \in \mathcal{E}$. In order to derive the transition probability function we note that the probability to take some transition $\ell \to \ell'$ $(\ell, \ell' \in \mathbb{L})$ with rate $R_{\ell,\ell'}(t)$ within Δt units of time at time t is given by:

$$\text{Prob}\{\ell \to \ell', t, \Delta t\} = \int_0^{\Delta t} R_{\ell,\ell'}(t + \tau) e^{-\int_0^\tau E_\ell(t+v)dv} d\tau. \tag{1}$$

As a next step, we rewrite Eq. (1) into:

$$\text{Prob}\{\ell \to \ell', t, \Delta t\} = \int_t^{t+\Delta t} R_{\ell,\ell'}(\tau) e^{-\int_t^\tau E_\ell(v)dv} d\tau. \tag{2}$$

It is not difficult to see that $\text{Prob}\{\ell \to \ell', t, \Delta t\}$ measures the probability to move from state $\xi = (\ell, t)$ to the set of states $A = \{(\ell', x) \mid x \in [t, t + \Delta t]\}$. For an arbitrary set of states A, Eq. (2) results in transition kernel:

$$\Pr(\xi, A) = \sum_{\ell' \in A_{\mathbb{L}}} \int_{A^{\mathbb{R}}_{\ell'} \cap [0,\infty[\oplus \xi_{\mathbb{R}}} R_{\xi_{\mathbb{L}},\ell'}(\tau) e^{-\int_{\xi_{\mathbb{R}}}^{\tau} E_{\xi_{\mathbb{L}}}(v)dv} d\tau, \tag{3}$$

where for any interval I (in our case $I = [0, \infty[$), $I \oplus \xi_{\mathbb{R}} = \{x + \xi_{\mathbb{R}} \mid x \in I\}$. Note that the domain of the integral in Eq. (3) is composed of the sets $A^{\mathbb{R}}_{\ell'}$ and $[0, \infty[\oplus\xi_{\mathbb{R}}$. The latter set ensures that the probability to jump back in time is zero. From Eq. (3) we directly obtain that the probability to move from state ξ to the set A of states in the interval I is given by:

$$\Pr(\xi, A, I) = \sum_{\ell' \in A_{\mathbb{L}}} \int_{A^{\mathbb{R}}_{\ell'} \cap I \oplus \xi_{\mathbb{R}}} R_{\xi_{\mathbb{L}},\ell'}(\tau) e^{-\int_{\xi_{\mathbb{R}}}^{\tau} E_{\xi_{\mathbb{L}}}(v)dv} d\tau. \tag{4}$$

Proposition 1. *The transition kernel is a probability measure provided:*

$$\lim_{\tau \to \infty} \int_{\xi_{\mathbb{R}}}^{\tau} E_{\xi_{\mathbb{L}}}(v)dv = \infty, \quad \text{for } \xi \in \mathbb{E}.$$

For every location $\ell \in \mathbb{L}$ the divergence of the integral from Proposition 1 or $\int_0^\infty E_\ell(v)dv$ can be tested by searching for a simpler function $h_\ell(v)$ such that $h_\ell(v) \leq E_\ell(v)$ for every $v \in [0, \infty[$ and for which we can easily show that $\int_0^\infty h_\ell(v)dv = \infty$.

Besides the transition kernel, for any ICTMC we can define the *transient probability distribution* which indicates the probability $\pi_j\,(t + \Delta t)$ to be in state j at time $t + \Delta t$:

$$\pi_j\,(t + \Delta t) = \sum_{i \in \mathbb{L}} \text{Prob}\,\{X(t) = i\} \cdot \text{Prob}\,\{X(t + \Delta t) = j | X(t) = i\}\,, \tag{5}$$

where $X(t)$ is a random variable indicating the location of the ICTMC at time t. Notice that $\text{Prob}\,\{X(t + \Delta t) = j | X(t) = i\}$ is a multi-step version of the transition kernel Pr as the number of transitions between states i and j can be arbitrary. For ICTMCs the transient behavior can also be described by a homogeneous system of ODEs (Chapman-Kolmogorov equations):

$$\frac{d\boldsymbol{\pi}(t)}{dt} = \boldsymbol{\pi}(t)\mathbf{Q}(t), \qquad \sum_{i=1}^{n} \pi_i(t_0) = 1, \tag{6}$$

where $\mathbf{Q}(t) = \mathbf{R}(t) - \mathbf{E}(t)$ is the *infinitesimal generator* and the vector $\boldsymbol{\pi}(t_0) = [\pi_1(t_0), \ldots, \pi_n(t_0)]$ is the initial condition.

Example 2. Fig. 2 depicts the transient probability distribution (see Eq. (5)) of states ℓ_0 and ℓ_3 from Fig. 1 for two cases: (1) a time-dependent rate function $\mu(t)$ with $\mu_{min} = 1$, $\mu_{max} = 2$ and $a = 3$, (2) a constant rate function $\mu(t) = 2$. For both cases $\lambda = 2$. Note the significant difference between the transient probabilities for these time-dependent and constant cases.

3 Continuous Hennessy-Milner Logic

Background. Hennessy-Milner Logic (HML) [11] is an action-based logic aimed at specifying properties of labeled transition systems. Its syntax is given by:

$$\Phi ::= \top \mid \Phi \wedge \Phi \mid \neg\Phi \mid \langle a \rangle \Phi,$$

where a is an action. The semantics is defined over process P. $P \models \langle a \rangle \Phi$ whenever for some process P', $P' \models \Phi$ and $P \xrightarrow{a} P'$.

Several probabilistic variants of HML exist. Larsen and Skou [18] have extended HML for discrete probabilistic systems by adding two new operators: Δ_a and $\langle a \rangle_p \Phi$ with $p \in [0,1] \cap \mathbb{Q}$. $P \models \Delta_a$ holds when $P \stackrel{a}{\nrightarrow}$ and $P \models \langle a \rangle_p \Phi$ holds when $P \xrightarrow{a}_\nu S$ (ν is the probability to move from P to the set of processes S) such that $\nu \geq p$ and $\forall s \in S.\, s \models \Phi$. Recently, Parma & Segala [19] defined HML for probabilistic automata [21]. Clark *et al.* [7] defined a similar variant as Larsen and Skou for action-labeled CTMCs [14] where the probability p in $\langle a \rangle_p \Phi$ is replaced by a rate.

Syntax and semantics. Our logic for ICTMCs is inspired by Larsen and Skou's variant of HML. We consider a state-based variant of HML and include a notion of time. The Continuous Hennessy-Milner Logic (CHML) for ICTMCs is defined by the following grammar:

Definition 2 (Syntax). *For* ICTMC $\mathcal{C}$ *with state space* $\mathbb{E}$*, atomic proposition* $a \in AP$*,* $p \in [0,1] \cap \mathbb{Q}$*, interval* $I \subseteq \mathbb{R}_{\geqslant 0}$ *with rational bounds and* $\trianglelefteq \in \{<, \leq, \geq, >\}$*, the grammar of* **CHML** *is:*

$$\Phi ::= \top \mid a \mid \Phi \wedge \Phi \mid \neg\Phi \mid \langle \Phi \rangle^{I}_{\trianglelefteq p}.$$

Here, the formula $\langle \Phi \rangle^{I}_{\trianglelefteq p}$ asserts that a state satisfying Φ can be reached within the interval I with probability within the threshold of p.

Example 3. Consider the ICTMC from Fig. 1 with the labels $L(\ell_0) = \mathit{empty}$, $L(\ell_1) = \#1$,$L(\ell_2) = \#2$ and $L(\ell_3) = \mathit{full}$. The formula $\langle \#2 \rangle^{[1,4]}_{\geqslant 0.3} \wedge \neg\#1$ holds in any state ξ with $L(\xi_{\mathbb{L}}) = \{\neg\#1\}$, which may jump in a single transition to the state ξ' such that $L(\xi'_{\mathbb{L}}) = \{\#2\}$ in the interval $[1,4]$ with probability at least 0.3.

Applying the negation operator $\neg$ to the operator $\langle\cdot\rangle$ yields:

$$\neg \left(\langle \Phi \rangle^{I}_{\leq p} \right) = \langle \neg\Phi \rangle^{I}_{\geq 1-p} \quad \text{and} \quad \neg \left(\langle \Phi \rangle^{I}_{> p} \right) = \langle \neg\Phi \rangle^{I}_{< 1-p}.$$

It is important to note that CHML can be viewed as a sub-logic of Continuous Stochastic Logic (CSL)[5] with $\langle \Phi \rangle^{I}_{\trianglelefteq p}$ being equivalent to the next operator $\mathcal{P}_{\trianglelefteq p}\left(X^I \Phi\right)$ of CSL. The substantial difference between CHML and CSL is that the satisfaction relation for any CHML-formula is defined over the set of states of an ICTMC which is *uncountable*. This difference is due to the fact that the evolution of the ICTMC depends on (a global notion of) time, whereas in CTMCs this is not the case. The global time in ICTMCs implies a continuous state-space rather than a finite one as in CTMCs. Finally, the definition of CHML is more similar to the approach of Desharnais [8] where CSL is defined for continuous-time Markov processes. As opposed to the approach of Desharnais, in this paper we are more interested in model checking. Let $[\![\Phi]\!] = \{\xi \in \mathbb{E} \mid \xi \models \Phi\}$ denote the set of states satisfying Φ, where $\models$ is defined as follows:

Definition 3 (Semantics). *The relation* $\models \subseteq \mathbb{E} \times$ **CHML** *is defined by:*

$$\begin{array}{lll} \xi \models \top & \textit{for all} & \xi \in \mathbb{E}, \\ \xi \models a & \textit{iff} & a \in L(\xi_{\mathbb{L}}), \\ \xi \models \Phi \wedge \Psi & \textit{iff} & \xi \models \Phi \textit{ and } \xi \models \Psi, \end{array} \qquad \begin{array}{lll} \xi \models \neg\Phi & \textit{iff} & \textit{not } \xi \models \Phi, \\ \xi \models \langle\Phi\rangle^{I}_{\unlhd p} & \textit{iff} & \mathrm{Pr}(\xi, [\![\Phi]\!], I) \unlhd p. \end{array}$$

In order for $\models$ to be well-defined, we need to address measurability.

Lemma 1. *For any formula* $\Phi \in$ **CHML** *the set* $[\![\Phi]\!]$ *is measurable.*

The ICTMC $\mathcal{C}$ satisfies formula Φ, denoted $\mathcal{C} \models \Phi$, iff $\xi_0 \in [\![\Phi]\!]$.

Bisimulation. It is well-known that strong bisimulation coincides with HML equivalence. In a similar vein, Larsen and Skou [18] showed that their logic characterizes probabilistic bisimulation. Recently, we have defined a notion of bisimulation for ICTMCs [10] which has the same coinductive flavor as in the case of CTMCs [6] and IMCs [13]. Our bisimulation is a structural notion and is defined on the level of the syntax of ICTMCs rather than their underlying infinite state space. Let $R(\ell, C, t)$ be the sum of all outgoing rates from location ℓ to the set C of locations at time t given by $\sum_i \{\!| \lambda(t) \mid \ell \xrightarrow{\lambda(t)}_i \ell'', \ell'' \in C \,|\!\}$, where $\ell \xrightarrow{\lambda(t)}_i \ell''$ denotes the i'th transition from location ℓ to location ℓ'' labeled with $\lambda(t)$ and $\{\!| \cdots |\!\}$ denotes a multi-set.

Definition 4 (Bisimulation). *An equivalence* $\mathcal{R} \subseteq \mathbb{L} \times \mathbb{L}$ *is a* bisimulation *whenever for all* $(\ell, \ell') \in \mathcal{R}$ *it holds that* $L(\ell) = L(\ell')$ *and* $R(\ell, C, t) = R(\ell', C, t)$ *for all* $t \in \mathbb{R}_{\geqslant 0}$ *and* $C \in \mathbb{L}/\mathcal{R}$. ℓ *and* ℓ' *are bisimilar, denoted* $\ell \sim \ell'$, *if* (ℓ, ℓ') *is contained in some bisimulation* $\mathcal{R}$.

In [5] there is a well-known result which states that bisimulation (for CTMCs) preserves the validity of CSL formulas. A similar result can be obtained also for HML first, by lifting the notion of bisimulation to the set of states $\mathbb{E}$ as follows:

Definition 5. *An equivalence* $\mathcal{R} \subseteq \mathbb{E} \times \mathbb{E}$ *is an* $\mathbb{E}$*-bisimulation whenever for all* $(\xi, \xi') \in \mathcal{R}$ *holds* $\xi_{\mathbb{L}} \sim \xi'_{\mathbb{L}}$ *and* $\xi_{\mathbb{R}} = \xi'_{\mathbb{R}}$. ξ *and* ξ' *are* $\mathbb{E}$*-bisimilar, denoted* $\xi \sim_{\mathbb{E}} \xi'$, *if* (ξ, ξ') *is contained in some* $\mathbb{E}$*-bisimulation* $\mathcal{R}$.

Note that the $\mathbb{E}$-bisimulation is not the coarsest one. As for a constant rate matrix one can define a bisimulation such that any two states ξ and ξ' with $\xi_{\mathbb{L}} = \xi'_{\mathbb{L}}$ and $\xi_{\mathbb{R}} \neq \xi'_{\mathbb{R}}$ will be bisimilar while $\xi \nsim_{\mathbb{E}} \xi'$. On the other hand it is not difficult to see that the conditions for $\mathbb{E}$-bisimilarity in Definition 5 are sufficient to ensure $(\mathbb{E}/{\sim_{\mathbb{E}}})_{\mathbb{L}} = \mathbb{L}/{\sim}$.

Theorem 1. *For any formula* $\Phi \in \mathbf{CHML}$, *(*$\xi \models \Phi$ *iff* $\xi' \models \Phi$*) iff* $\xi \sim_{\mathbb{E}} \xi'$.

Due to this theorem, any verification results on $\mathcal{C}/\!\sim$, the quotient of $\mathcal{C}$ under $\sim$, carries over to $\mathcal{C}$ since $\mathcal{C}/\!\sim$ is bisimilar to $\mathcal{C}$. A bisimulation minimization algorithm for ICTMCs with piecewise constant rate functions has recently been proposed [10] and requires $\mathcal{O}\left(Nm \log n\right)$ time, where N is the number of constant pieces of the rate matrix $\mathbf{R}(t)$, and m, n are the number of transitions and locations of $\mathcal{C}$, respectively.

4 Model Checking Continuous Hennessy-Milner Logic

Continuous Hennessy-Milner Logic describes properties which can be verified for every state of $\mathbb{E}$. When one attempts to develop model-checking algorithms for CHML one has to consider that the state space $\mathbb{E}$ is in fact continuous (i.e., consists of uncountably many states). This is a main difference with logics for CTMCs, such as CSL, where the state space is denumerable since the behavior of a CTMC only depends on the current location and not on the amount of time spent there. Therefore, our aim is to group all states $\xi \in \mathbb{E}$, where $\xi \models \Phi$ into tuples $(\ell, \mathcal{I})$ with $\ell \in \mathbb{L}$ and $\mathcal{I} \subseteq \mathbb{R}_{\geqslant 0}$ - formed of a finite union of intervals:

$$[\![\Phi]\!] := \{(\ell, \mathcal{I}) \mid \ell \in \mathbb{L}, \mathcal{I} = \{\xi_{\mathbb{R}} \mid \xi \in \mathbb{E}, \xi \models \Phi, \xi_{\mathbb{L}} = \ell\}\} \setminus \{(\ell, \emptyset) \mid \ell \in \mathbb{L}\}\,.$$

Using the tuple (location-interval) representation we can form the satisfaction set of any propositional formula $\Phi \in \mathbf{CHML}$ as:

$$\begin{aligned} [\![\,\top\,]\!] &= \{(\ell, \mathbb{R}_{\geqslant 0}) \mid \ell \in \mathbb{L}\} \quad [\![\,a\,]\!] = \{(\ell, \mathbb{R}_{\geqslant 0}) \mid \ell \in \mathbb{L}, a \in L(\ell)\} \\ [\![\,\Phi \wedge \Psi\,]\!] &= \left\{\left(\ell, [\![\,\Phi\,]\!]^{\mathbb{R}}_{\ell} \cap [\![\,\Psi\,]\!]^{\mathbb{R}}_{\ell}\right) \mid \ell \in \mathbb{L}, \ell \in [\![\,\Phi\,]\!]_{\mathbb{L}} \cap [\![\,\Psi\,]\!]_{\mathbb{L}}\right\} \\ [\![\,\neg\Phi\,]\!] &= \{(\ell, \mathbb{R}_{\geqslant 0}) \mid \ell \in \mathbb{L} \backslash [\![\,\Phi\,]\!]_{\mathbb{L}}\} \cup \left\{\left(\ell, \mathbb{R}_{\geqslant 0} \backslash [\![\,\Phi\,]\!]^{\mathbb{R}}_{\ell}\right) \mid \ell \in [\![\,\Phi\,]\!]_{\mathbb{L}}\right\} \end{aligned}$$

As every element of the set $[\![\Phi]\!]$ is a tuple $(\ell, \mathcal{I})$ the intersection is done component-wise, i.e., per location ℓ and component $\mathcal{I}$.

Example 4. Consider the ICTMC from Fig. 1(a) and the sets $[\![\,\Phi\,]\!] = \{(\ell_0, [1, 2])\}$, $[\![\,\Psi\,]\!] = \{(\ell_3, [0, 5] \cup [8, \infty[)\}$. We have the following satisfaction sets:

$$\begin{aligned} [\![\,\neg\Phi\,]\!] &= \{(\ell_0, [0, 1[\cup]2, \infty[), (\ell_1, \mathbb{R}_{\geqslant 0}), (\ell_2, \mathbb{R}_{\geqslant 0}), (\ell_3, \mathbb{R}_{\geqslant 0})\} \\ [\![\,\neg\Psi\,]\!] &= \{(\ell_0, \mathbb{R}_{\geqslant 0}), (\ell_1, \mathbb{R}_{\geqslant 0}), (\ell_2, \mathbb{R}_{\geqslant 0}), (\ell_3,]5, 8[)\} \\ [\![\,\neg\Phi \wedge \neg\Psi\,]\!] &= \{(\ell_0, [0, 1[\cup]2, \infty[), (\ell_1, \mathbb{R}_{\geqslant 0}), (\ell_2, \mathbb{R}_{\geqslant 0}), (\ell_3,]5, 8[)\}\,. \end{aligned}$$

By using only the above four cases (initially we don't consider the formula $\langle\Phi\rangle^{I}_{\trianglelefteq p}$) it is not difficult to see that every set $[\![\Phi]\!]$ will be formed of finitely many tuples $(\ell, \mathcal{I})$ where $\mathcal{I} \subseteq \mathbb{R}_{\geqslant 0}$. The most challenging part is to find all tuples (elements) $(\ell, \mathcal{I})$ of the set $[\![\,\langle\Phi\rangle^{I}_{\trianglelefteq p}\,]\!]$.

Verifying $\langle\Phi\rangle^{I}_{\trianglelefteq p}$-formulas. Using Eq. (4), the set $[\![\langle\Phi\rangle^{I}_{\trianglelefteq p}]\!]$ for any $\Phi \in$ **CHML** is given by:

$$[\![\langle\Phi\rangle^{I}_{\trianglelefteq p}]\!] = \{\xi \in \mathbb{E} \mid \Pr(\xi, [\![\Phi]\!], I) \trianglelefteq p\} \quad \text{where} \tag{7}$$

$$\Pr(\xi, [\![\Phi]\!], I) = \sum_{\ell' \in [\![\Phi]\!]_{\mathbb{L}}} \int_{[\![\Phi]\!]^{\mathbb{R}}_{\ell'} \cap I \oplus \xi_{\mathbb{R}}} R_{\xi_{\mathbb{L}},\ell'}(\tau) e^{-\int_{\xi_{\mathbb{R}}}^{\tau} E_{\xi_{\mathbb{L}}}(v)dv} d\tau. \tag{8}$$

Here our task is to group all $\xi \in [\![\langle\Phi\rangle^{I}_{\trianglelefteq p}]\!]$ into tuples $(\ell, \mathcal{I})$ such that $\ell \in [\![\langle\Phi\rangle^{I}_{\trianglelefteq p}]\!]_{\mathbb{L}}$ and $\mathcal{I} = [\![\langle\Phi\rangle^{I}_{\trianglelefteq p}]\!]^{\mathbb{R}}_{\ell}$. For every $\ell \in \mathbb{L}$, a two-step procedure corresponding to equations (7) and (8) follows:

1. Find the set $\mathcal{X}$ of solutions by solving the equation (recall that $\xi = (\ell, x)$)
$$\Pr((\ell, x), [\![\Phi]\!], I) = p, \tag{9}$$
where $x \in \mathbb{R}_{\geqslant 0}$ is the unknown variable and
2. Find $\mathcal{I}$ by using $\mathcal{X}$ such that $\Pr((\ell, x^*), [\![\Phi]\!], I) \trianglelefteq p$ and $x^* \in \mathcal{I}$.

The second step is straightforward i.e., after computing the set $\mathcal{X}$ the interval $\mathcal{I}$ is computed by checking the condition $\Pr((\ell, x^*), [\![\Phi]\!], I) \trianglelefteq p$ for every sequential pair of solutions $x_i, x_{i+1} \in \mathcal{X}$ such that $x_i < x_{i+1}$ and $x^* = \frac{x_i + x_{i+1}}{2}$. The first step is a bit more problematic. The difficulty lies in computing the integral from Eq. (8) over a time-variant domain $[\![\Phi]\!]^{\mathbb{R}}_{\ell'} \cap I \oplus \xi_{\mathbb{R}}$ of a time-variant rate $R_{\xi_{\mathbb{L}},\ell'}(\tau)$ and exit rate $E_{\xi_{\mathbb{L}}}(v)$. Moreover, we aim to obtain a finite set of solutions $\mathcal{X}$. In order to meet this challenge, we impose some conditions on the rate functions of ICTMCs.

We assume that the rates $R_{\ell,\ell'}(\tau)$ for any two locations ℓ and ℓ' are piecewise constant functions which are *right-continuous with left limits*. Formally, this means that $R_{\ell,\ell'}(\tau) = R^{(k)}_{\ell,\ell'}$ for every $\tau \in [t_k, t_{k+1}[$, where $t_{k+1} = t_k + \Delta t$ (here Δt is the time discretization parameter), $t_{N+1} = \infty$, $k \in \{1, \cdots, N\}$ and N is the total number of constant pieces. We thus obtain $E_{\ell}(\tau) = E^{(k)}_{\ell}$ for every $\tau \in [t_k, t_{k+1}[$. Notice that the restriction of rates to be right-continuous with left limits is not crucial as the values of the rates at discrete points are not relevant. This partition of the time-axis $\mathbb{R}_{\geqslant 0}$ will ensure that for every tuple $(\ell, \mathcal{I})$ in $[\![\langle\Phi\rangle^{I}_{\trianglelefteq p}]\!]$ the component $\mathcal{I}$ will consist of a finite union of disjoint intervals. In fact, later on it will be shown that for piecewise constant rate functions the set $\mathcal{X}$ is finite. Now consider the CHML-formula Φ from $[\![\langle\Phi\rangle^{I}_{\trianglelefteq p}]\!]$. Assume that in every tuple $(\ell', \mathcal{I})$ from $[\![\Phi]\!]$, $\mathcal{I} = [\![\Phi]\!]^{\mathbb{R}}_{\ell'}$ is a finite union of disjoint intervals i.e., $[\![\Phi]\!]^{\mathbb{R}}_{\ell'} = \biguplus_{i=1}^{\theta} \mathcal{I}^{(i)}_{\ell'}$, where θ is the total number of such intervals. Eq. (8) becomes:

$$\Pr((\ell, x), [\![\Phi]\!], I) = \sum_{\ell' \in [\![\Phi]\!]_{\mathbb{L}}} \sum_{i=1}^{\theta} \int_{\mathcal{I}^{(i)}_{\ell'} \cap I \oplus x} R_{\ell,\ell'}(\tau) e^{-\int_{x}^{\tau} E_{\ell}(v)dv} d\tau. \tag{10}$$

It is important to note that the intervals $\mathcal{I}^{(i)}_{\ell'}$ can be open, closed, or half-closed. Therefore, for the integral in Eq. (10) one has to consider the limit from the

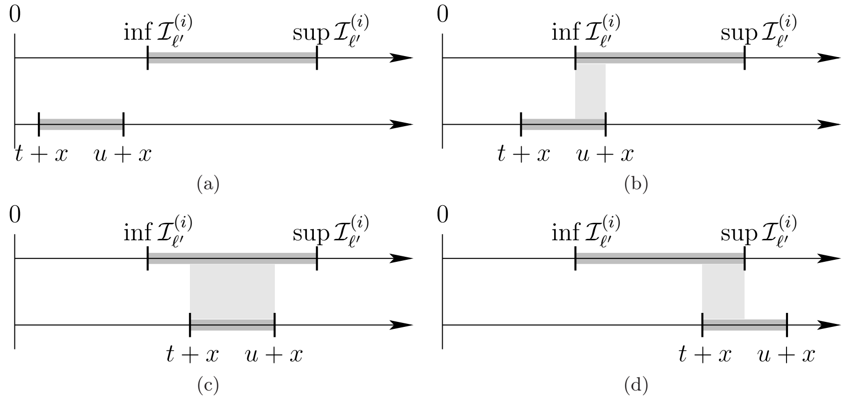


Fig. 3. The position of $\mathcal{I}^{(i)}_{\ell'}$ and $[t,u]\oplus x$ together with their intersection in (gray) for the case $\sup \mathcal{I}^{(i)}_{\ell'} - \inf \mathcal{I}^{(i)}_{\ell'} > u-t$.

right, as well as from the left. In our case this is not necessary as all rates are right-continuous with left limits.

Until now we set the basis for solving Eq. (9) by considering a finite partition of the time-axis. The final step is to compute the integral from Eq. (10) over the time-variant domain $\mathcal{I}^{(i)}_{\ell'} \cap I \oplus x$ (by varying x the size of the integration domain changes). For this we take $I = [t,u]$ and as a result the integration domain $\mathcal{I}^{(i)}_{\ell'} \cap [t,u] \oplus x$ takes the form:

$$\mathcal{I}^{(i)}_{\ell'} \cap [t,u] \oplus x = \left[\max\left\{\inf \mathcal{I}^{(i)}_{\ell'}, t+x\right\}, \min\left\{\sup \mathcal{I}^{(i)}_{\ell'}, u+x\right\}\right]. \tag{11}$$

Notice the interval in Eq. (11) is not necessary closed as its bounds depend on the interval $\mathcal{I}^{(i)}_{\ell'}$. For instance, if $\inf \mathcal{I}^{(i)}_{\ell'} > t+x$ and $\mathcal{I}^{(i)}_{\ell'}$ is left-open, then the integration domain will be also left-open.

The interval in Eq. (11) strongly depends on the position of $\mathcal{I}^{(i)}_{\ell'}$ relative to $[t,u] \oplus x$. This means that there are several configurations given by $\sup \mathcal{I}^{(i)}_{\ell'}$, $\inf \mathcal{I}^{(i)}_{\ell'}$, t, and u. For instance, Fig. 3 depicts the position of $\mathcal{I}^{(i)}_{\ell'}$ and $[t,u] \oplus x$ when $\sup \mathcal{I}^{(i)}_{\ell'} - \inf \mathcal{I}^{(i)}_{\ell'} > u-t$. As you can see, the relative placement of both intervals in Fig. 3 is crucial for the computation of the integral from Eq. (10).

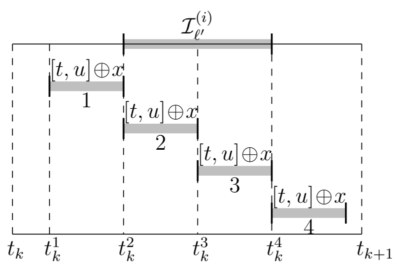


Fig. 4. Time point events.

At the beginning of the section we have assumed that the rates are piecewise constant or more intuitively we have discretized the time-axis into intervals $[t_k, t_{k+1}[$ during which the rates are constant. This discretization gives us the possibility to find a closed-form expression for Eq. (10) that enables us to solve Eq. (9). The derived expression will not contain the integral, but instead will be a linear combination of exponential functions as we will show below. Now by considering each interval $[t_k, t_{k+1}[$ separately we can derive an expression for $\Pr((\ell, x), [\![\Phi]\!], I)$ by computing the integral from Eq. (10) with the condition that $x \in [t_k, t_{k+1}[$. As was shown before, the computation of the integral strongly depends on the integration domain. By varying x from t_k up to t_{k+1}, the interval $[t, u] \oplus x$ shifts as shown in Fig. 3. Note that different values of x mark the beginning of time when the intersection between $\mathcal{I}_{\ell'}^{(i)}$ and $[t, u] \oplus x$ will be empty or not. There are four of such time points (see Fig. 4) for the case $\sup \mathcal{I}_{\ell'}^{(i)} - \inf \mathcal{I}_{\ell'}^{(i)} > u - t$:

1. $t_k^1 = x$ - is the moment of time when $u + x \geq \inf \mathcal{I}_{\ell'}^{(i)}$,
2. $t_k^2 = x$ - is the moment of time when $t + x \geq \inf \mathcal{I}_{\ell'}^{(i)}$ and $[t, u] \oplus x \subseteq \mathcal{I}_{\ell'}^{(i)}$,
3. $t_k^3 = x$ - is the moment of time when $u + x \geq \sup \mathcal{I}_{\ell'}^{(i)}$,
4. $t_k^4 = x$ - is the moment of time when $t + x \geq \sup \mathcal{I}_{\ell'}^{(i)}$ and $\mathcal{I}_{\ell'}^{(i)} \cap [t, u] \oplus x = \emptyset$ when $t + x > \sup \mathcal{I}_{\ell'}^{(i)}$.

Note that in order to simplify the notations we do not indicate the indices ℓ' and i to t_k^j as it is clear that every t_k^j, $j \in \{1, \ldots, 4\}$ is computed respective to the interval $\mathcal{I}_{\ell'}^{(i)}$ i.e., for every ℓ' and i, the time point t_k^j will be different. Also if some time point t_k^j is not defined then $t_k^j = t_k$.

Using t_k^j we divide the interval $[t_k, t_{k+1}[$ into five sub-intervals $[t_k, t_k^1[$, $[t_k^1, t_k^2[$, $[t_k^2, t_k^3[$, $[t_k^3, t_k^4[$ and $[t_k^4, t_{k+1}[$. For each of the mentioned sub-intervals we can compute the integral from Eq. (10). Here we only consider the case when $k = N$ and $\sup \mathcal{I}_{\ell'}^{(i)} - \inf \mathcal{I}_{\ell'}^{(i)} > u - t$. For the remaining cases, the procedure is straightforward. Distinguish (as indicated above), the following five cases:

- $x \in [t_N, t_N^1[\; \Rightarrow \mathcal{I}_{\ell'}^{(i)} \cap [t, u] \oplus x = \emptyset$.
- $x \in [t_N^1, t_N^2[\Rightarrow \max \left\{ \inf \mathcal{I}_{\ell'}^{(i)}, t + x \right\} = \inf \mathcal{I}_{\ell'}^{(i)}, \min \left\{ \sup \mathcal{I}_{\ell'}^{(i)}, u + x \right\} = u + x$,

$$\int_{\inf \mathcal{I}_{\ell'}^{(i)}}^{u+x} R_{\ell,\ell'}(\tau) e^{-\int_x^\tau E_\ell(v) dv} d\tau = \frac{R_{\ell,\ell'}^{(N)}}{E_\ell^{(N)}} \left(e^{-\left(\inf \mathcal{I}_{\ell'}^{(i)} - x\right) E_\ell^{(N)}} - e^{-u E_\ell^{(N)}} \right).$$

- $x \in [t_N^2, t_N^3[\Rightarrow \max \left\{ \inf \mathcal{I}_{\ell'}^{(i)}, t + x \right\} = t + x, \min \left\{ \sup \mathcal{I}_{\ell'}^{(i)}, u + x \right\} = u + x$,

$$\int_{t+x}^{u+x} R_{\ell,\ell'}(\tau) e^{-\int_x^\tau E_\ell(v) dv} d\tau = \frac{R_{\ell,\ell'}^{(N)}}{E_\ell^{(N)}} \left(e^{-t E_\ell^{(N)}} - e^{-u E_\ell^{(N)}} \right).$$

- $x \in [t_N^3, t_N^4[\Rightarrow \max \left\{ \inf \mathcal{I}_{\ell'}^{(i)}, t + x \right\} = t + x, \min \left\{ \sup \mathcal{I}_{\ell'}^{(i)}, u + x \right\} = \sup \mathcal{I}_{\ell'}^{(i)}$,

$$\int_{t+x}^{\sup \mathcal{I}_{\ell'}^{(i)}} R_{\ell,\ell'}(\tau) e^{-\int_x^\tau E_\ell(v) dv} d\tau = \frac{R_{\ell,\ell'}^{(N)}}{E_\ell^{(N)}} \left(e^{-t E_\ell^{(N)}} - e^{-\left(\sup \mathcal{I}_{\ell'}^{(i)} - x\right) E_\ell^{(N)}} \right).$$

- $x \in [t_N^4, \infty[\; \Rightarrow \mathcal{I}_{\ell'}^{(i)} \cap [t, u] \oplus x = \emptyset$.

For all five intervals the integral in Eq. (10) has the same form given by the expression $a^{i,j}_{N,\ell'} + b^{i,j}_{N,\ell'} e^{xE^{(N)}_{\ell}}$ (it is in fact a linear combination of exponential functions). The constants $a^{i,j}_{N,\ell'}$ and $b^{i,j}_{N,\ell'}$ corresponding to the interval $[t^j_N, t^{j+1}_N[$, $j \in \{1,2,3\}$ are formed by the expressions containing $R^{(N)}_{\ell,\ell'}$ and $E^{(N)}_{\ell}$. Given $a^{i,j}_{N,\ell'} + b^{i,j}_{N,\ell'} e^{xE^{(N)}_{\ell}}$ for each interval $[t^j_N, t^{j+1}_N[$, Eq. (10) can be simplified to:

$$\Pr((\ell,x), \llbracket\Phi\rrbracket, I) = \sum_{\ell' \in \llbracket\Phi\rrbracket_{\mathbb{L}}} \sum_{i=1}^{\theta} \left(a^{i,j}_{N,\ell'} + b^{i,j}_{N,\ell'} e^{xE^{(N)}_{\ell}} \right) \chi^{i,j}_{\ell'}(x), \tag{12}$$

where $\chi^{i,j}_{\ell'}(x) = 1$ for every $x \in [t^j_N, t^{j+1}_N[$ and 0 otherwise. Note that there is at most one solution (i.e., $|\mathcal{X}| \leq 1$) when solving Eq. (9) using Eq. (12) (i.e., the case when $k = N$).

Now we proceed with the interval $[t_k, t_{k+1}[$ for $k < N$. Here the general form of Eq. (10) becomes more complex as transition rates are not constant on the interval of time $[0, t_N[$. We obtain the following result, which is a major stepping-stone towards a model-checking algorithm for piecewise constant ICTMCs.

Theorem 2. *For any $x \in [t_k, t_{k+1}[$ and $k < N$, Eq. (10) takes the general form:*

$$\Pr((\ell,x), \llbracket\Phi\rrbracket, I) = a^{(1)}_k e^{xb_1} + a^{(2)}_k e^{xb_2} + a^{(3)}_k e^{xb_3} + a^{(4)}_k e^{xb_4} + a^{(5)}_k e^{xb_5} + a^{(6)}_k, \tag{13}$$

where $a^{(i)}_k$ for $i = 1, \cdots, 6$ is constant and $b_1 = E^{(k)}_{\ell}$, $b_j = \left(b_1 - E^{(k_{j-1})}_{\ell}\right)$, for $j > 1$ with $k_1 = \left\lfloor \frac{t+t_k}{\Delta t} \right\rfloor + 1$, $k_2 = k_1 + 1$, $k_3 = \left\lfloor \frac{u+t_k}{\Delta t} \right\rfloor + 1$, $k_4 = k_3 + 1$.

Eq. (13) will be derived for every sub-interval $[t^j_k, t^{j+1}_k[$, $[t_k, t^1_k[$, and $[t^4_k, t_{k+1}[$, $j \in \{1,2,3\}$ also for the special case of the *two* intervals obtained from each $[t^1_k, t^2_k[$, $[t^2_k, t^3_k[$, and $[t^3_k, t^4_k[$. Actually for every $\mathcal{I}^{(i)}_{\ell'}$ the interval $[t_k, t_{k+1}[$ will be partitioned into at most eight sub-intervals on which different derivations of Eq. (13) will be obtained.

Example 5. Let us consider an ICTMC $\mathcal{C}$ with the set of locations $\mathbb{L} = \{\ell, \ell'\}$, where ℓ is the initial location of $\mathcal{C}$. There is a transition $\ell \to \ell'$ with rate $R_{\ell,\ell'}(\tau)$ and an exit rate $E_{\ell}(\tau)$ for location ℓ defined as:

1. $R_{\ell,\ell'}(\tau) = R^{(1)}_{\ell,\ell'}$, $E_{\ell}(\tau) = E^{(1)}_{\ell}$ when $\tau \in [0,3[$ and
2. $R_{\ell,\ell'}(\tau) = R^{(2)}_{\ell,\ell'}$, $E_{\ell}(\tau) = E^{(2)}_{\ell}$ when $\tau \in [3,\infty[$,

$R^{(1)}_{\ell,\ell'}$, $R^{(2)}_{\ell,\ell'}$, $E^{(1)}_{\ell}$ and $E^{(2)}_{\ell}$ are constant. Here notice $\Delta t = 3$ and $N = 2$.

Assume we have the formula $\langle a \rangle^{[0,\infty[}_{\trianglelefteq p}$, where $I = [0,\infty[$, $L(\ell') = a$, $a \in AP$ and $p \in [0,1]$. We want to find an expression for $\Pr((\ell,x), \llbracket a\rrbracket, [0,\infty[)$. Notice that in Eq. (10) $\llbracket a\rrbracket = \{(\ell', [0,\infty[)\}$, $\llbracket a\rrbracket_{\mathbb{L}} = \ell'$, $\theta_{\ell'} = 1$ and $\mathcal{I}^{(1)}_{\ell'} = [0,\infty[$. We consider two intervals $[t_1, t_2[= [0,3[$ and $[t_2, \infty[= [3,\infty[$. First we take $x \in [0,3[$ with

$t_1^2 = t_1^3 = 0$, $\max\left\{\inf \mathcal{I}_{\ell'}^{(i)}, t+x\right\} = t+x = x$, $\min\left\{\sup \mathcal{I}_{\ell'}^{(i)}, u+x\right\} = \infty$. We get that $\Pr((\ell, x), [\![a]\!], [0, \infty[) =$

$$\int_x^{\infty} R_{\ell,\ell'}(\tau) e^{-\int_x^{\tau} E_\ell(v)dv} d\tau = \int_x^3 R_{\ell,\ell'}^{(1)} e^{-\int_x^{\tau} E_\ell^{(1)} dv} d\tau + \int_3^{\infty} R_{\ell,\ell'}^{(2)} e^{-\int_x^{\tau} E_\ell^{(2)} dv} d\tau =$$
$$\frac{R_{\ell,\ell'}^{(1)}}{E_\ell^{(1)}} \left(1 - e^{(x-3)E_\ell^{(1)}}\right) + \frac{R_{\ell,\ell'}^{(2)}}{E_\ell^{(2)}} e^{(x-3)E_\ell^{(2)}}.$$

Now we take $x \in [3, \infty[$ with $\max\left\{\inf \mathcal{I}_{\ell'}^{(i)}, t+x\right\} = x$, $\min\left\{\sup \mathcal{I}_{\ell'}^{(i)}, u+x\right\} = \infty$, $t_1^2 = t_1^3 = 3$ and we obtain

$$\Pr((\ell, x), [\![a]\!], [0, \infty[) = \int_x^{\infty} R_{\ell,\ell'}(\tau) e^{-\int_x^{\tau} E_\ell(v)dv} d\tau = \int_x^{\infty} R_{\ell,\ell'}^{(2)} e^{-\int_x^{\tau} E_\ell^{(2)} dv} d\tau = \frac{R_{\ell,\ell'}^{(2)}}{E_\ell^{(2)}}.$$

Using Theorem 2, we can solve Eq. (9) for every interval $[t_k, t_{k+1}[$ and $x \in [t_k, t_{k+1}[$ by means of:

$$a_k^{(1)} e^{xb_1} + a_k^{(2)} e^{xb_2} + a_k^{(3)} e^{xb_3} + a_k^{(4)} e^{xb_4} + a_k^{(5)} e^{xb_5} + a_k^{(6)} - p = 0. \tag{14}$$

Like in Eq. (12) the values of $a_k^{(j)}$, $j \in \{1, \ldots, 6\}$ are formed by the expressions containing $R_{\ell,\ell'}^{(k)}$ and $E_\ell^{(k)}$. The following theorem of Laguerre [17] provides an interesting property about the number of real solutions (zeros) of Eq. (14). For any sequence $a_1, \cdots, a_V$ let $W(a_1, \cdots, a_V)$ denote the number of sign changes in $a_1, \cdots, a_V$ defined by the number of pairs a_{m-i}, a_m $(m \geq 1)$ such that $a_{m-v} a_m < 0$ and $a_{m-v} = 0$ for $v = 1, \cdots, i-1$

Theorem 3 (Laguerre [17]). *Let $f(x) = \sum_{i=1}^{V} a_i e^{xb_i}$ be an exponential polynomial, where $a_i, b_i \in \mathbb{R}$ and $b_1 < \cdots < b_V$. The number $Z(f)$ of real zeros of f is bounded by $Z(f) \leq W(a_1, \cdots, a_V)$, and $Z(f)$ is of the same parity as $W(a_1, \cdots, a_V)$.*

From Laguerre's theorem it follows that the number of zeros of Eq. (14) is bounded by *five*. Laguerre's theorem does however neither provide a recipe for obtaining the solutions nor the intervals containing the solution. To obtain an algorithmic way to compute the zeros, we transform the exponential polynomial in Eq. (14) to the equivalent polynomial representation $P(z) = \sum_{i=1}^{6} c_i z^{n_i}$, where $n_1 > n_2 > n_3 > n_4 > n_5 > n_6$, n_1 - degree of P. Notice that the polynomial $P(z)$ can always be obtained because $b_i \in \mathbb{Q}_{\geqslant 0}$ in Eq. (14) can be represented as $b_i = m_i 10^{d_i}$ where $m_i, d_i \in \mathbb{Z}$. Therefore, transforming all $e^{xm_i 10^{d_i}}$'s to a common d_i and changing e^x to z yields $P(z)$.

Definition 6 (Sturm sequence). *Let $P(z)$ be a square-free (every root has multiplicity one) polynomial and $P'(z)$ denote its derivative. The* Sturm sequence *of $P(z)$ is the sequence $\{F_i(z)\}$ of polynomials defined by $F_0(z) = P(z)$, $F_1(z) = P'(z)$ and $F_i(z) = -\mathrm{rem}(F_{i-2}(z), F_{i-1}(z))$ for $i > 1$, where* $\mathrm{rem}(F_{i-1}(z), F_{i-2}(z))$ *is the remainder obtained by dividing $F_{i-2}(z)$ by $F_{i-1}(z)$.*

Notice if $P(z)$ is not square-free one can easily transform it to a square-free polynomial by computing the greatest common divisor of $P(z)$ and $P'(z)$.

Theorem 4 ([12]). *The number of real zeros of $P(z)$ in any interval $]a, b[$ is given by $W(F_0(a), F_1(a), \cdots, F_k(a)) - W(F_0(b), F_1(b), \cdots, F_k(b))$.*

Using the Sturm sequence we get the following algorithm which finds all real zeros $z_1, \cdots, z_m$ $(m \leq 5)$ of $P(z)$ in the interval $]a, b[$ with precision $\epsilon = 2^{-\mu}$. The

Algorithm 1 Polynomial solver

Require: polynomial $P(z)$, interval $]a, b[$ and precision $\epsilon = 2^{-\mu}$
Ensure: $z_1, \cdots, z_m$
1: $P'(z) :=$ **derivative** $(P(z))$
2: $\hat{P}(z) :=$ **gcd** $(P(z), P'(z))$
3: $\hat{P}'(z) :=$ **derivative** $\left(\hat{P}(z)\right)$
4: $\{F_0(z), F_1(z), \cdots, F_k(z)\} :=$ **Sturm** $\left(\hat{P}(z), \hat{P}'(z)\right)$
5: $\{]a_1, b_1[, \cdots,]a_m, b_m[\} :=$ **Binarysearch** $(\{F_0(z), F_1(z), \cdots, F_k(z)\},]a, b[)$
6: $\{z_1, \cdots, z_m\} :=$ **Newton** $(\{]a_1, b_1[, \cdots,]a_m, b_m[\}, \epsilon)$
7: **return** $\{z_1, \cdots, z_m\}$

above algorithm uses several functions. The *gcd* function computes the greatest common divisor of the polynomials $P(z)$ and $P'(z)$. The function *Binarysearch* divides the interval $]a, b[$ into subintervals $]a_i, b_i[$ using the bisection method [9] such that $z_i \in]a_i, b_i[$. Finally, the function *Newton* finds the approximate root z_i of $P(z)$ from $]a_i, b_i[$ with precision $\epsilon = 2^{-\mu}$, $\mu \in \mathbb{N}$ using the Newton method [9]. The first two lines in Alg. 1 are used to obtain a square free polynomial $\hat{P}(z)$.

Lemma 2. *The time complexity of Algorithm 1 is:*

$$\mathcal{O}\left(n_1^2 \log^2 n_1 (\log n_1 + s) + n_1 \log^2 n_1 \left|\log(\Delta t)\right| + n_1 \log \mu\right),$$

where n_1 is the degree of $P(z)$, s is the size in bits of the coefficients of $P(z)$ in the ring of integers, Δt is the time discretization parameter and μ is the number of bits-precision for the Newton method.

Proof. The running time of line 1 and 3 is $\mathcal{O}(n_1)$. The *gcd* (line 2) and the Sturm sequence (line 4) can be computed in $\mathcal{O}\left(n_1 \log^2 n_1\right)$ time [1]. Note that the minimal distance between any two zeros [20] of $P(z)$ is bounded from below by $2^{-\frac{n_1+2}{2}\log n_1 - sn_1 + s}$. Therefore, we get that the search-depth of *Binarysearch* is of order $\mathcal{O}(|\log(b-a)| + n_1 \log n_1 + sn_1)$. As every iteration of *Binarysearch* requires $\mathcal{O}\left(n_1 \log^2 n_1\right)$ time and by taking $b - a \leq \Delta t$, we get the running time of line 5 is $\mathcal{O}\left(n_1^2 \log^2 n_1 (\log n_1 + s) + n_1 \log^2 n_1 \left|\log(\Delta t)\right|\right)$. The Newton method takes $\mathcal{O}(n_1 \log \mu)$ time as there are in total $\mathcal{O}(\log \mu)$ iterations and each iteration requires $\mathcal{O}(n_1)$ time. The final time-complexity is obtained by combining the running-times of all functions in Algorithm 1 and the fact that $m \leq 5$.

Assume that θ (number of intervals of $[\![\Phi]\!]_{\ell'}^{\mathbb{R}}$) in Eq. (10) is bounded by M.

Lemma 3. *For every tuple* $(\ell, \mathcal{I})$ *of* $[\![\langle\Phi\rangle^{I}_{\trianglelefteq p}]\!]$ *and interval* $[t_k, t_{k+1}[$ *such that* $\mathcal{I} \subseteq [t_k, t_{k+1}[$, $\mathcal{I}$ *is given by a union of at most* $21nM+3$ *disjoint intervals where* n *is the number of locations.*

Proof. We already know that for every $\mathcal{I}^{(i)}_{\ell'}$ the interval $[t_k, t_{k+1}[$ will partitioned into a maximum of eight sub-intervals. From Eq. (10) we get that the total number of sub-intervals is $8nM$. Taking the intersection (due to the double summation in Eq. (10)) of all $8nM$ sub-intervals we obtain a smaller set of $8 + \sum_{j=1}^{nM-1}(8-1) = 7nM + 1$ sub-intervals on which we have to solve Eq. (14). By solving Eq. (14) or its equivalent Eq. (9) we get that for every $\mathcal{I}$, such that $\Pr((\ell, x), [\![\Phi]\!], I) \trianglelefteq p$ and $x \in \mathcal{I}$, will be formed of maximum three intervals (due to a bound of five for the number of zeros in Eq. (14)). Therefore, for every tuple $(\ell, \mathcal{I})$ of $[\![\langle\Phi\rangle^{I}_{\trianglelefteq p}]\!]$, $\mathcal{I}$ will be a disjoint union of at most $21nM + 3$ intervals.

We complete this section by addressing the time-complexity of the CHML model checking of ICTMCs. Now we take the formula $\Phi = \langle \ldots \langle a\rangle^{I_1}_{\trianglelefteq p_1} \ldots \rangle^{I_h}_{\trianglelefteq p_h}$ (without conjuction operator), where h is the nesting level of Φ and $a \in AP$. It is clear that every component $\mathcal{I}$ such that the tuple $(\ell, \mathcal{I})$ is in the set $[\![\Phi]\!]$, will be a disjoint union of $\mathcal{O}\left(21^h n^h\right)$ intervals.

Theorem 5. *The time complexity of model-checking* CHML*-formula* Φ *with nesting level* h *on an* ICTMC *with a piecewise constant rate matrix (N pieces):*

$$\mathcal{O}\left(21^h n^h N \cdot \left(n_1^2 \log^2 n_1 (\log n_1 + s) + n_1 \log^2 n_1 \left|\log(\Delta t)\right| + n_1 \log \mu + h \log n\right)\right),$$

where n *is the total number of locations,* n_1 *is the bound for the polynomial degree,* s *is the size in bits of the coefficients of polynomials in the ring of integers,* Δt *is the time discretization parameter and* μ *is the number of bits-precision for the Newton method.*

Proof. The theorem follows from Lemma 2 and 3. Also, we include the time complexity $\mathcal{O}\left(21^h n^h h \log n\right)$ of sorting the bounds of all $\mathcal{O}\left(21^h n^h\right)$ intervals.

5 Concluding Remarks

This paper presented a stochastic variant of Hennessy-Milner logic for inhomogeneous continuous-time Markov chains, and introduced an approximative verification algorithm for the setting in which rates are piecewise constant functions. Moreover, we have shown that the complexity of the model checking algorithm is exponential in the nesting depth of the formula and linear in the size of the ICTMC. Currently CHML is limited to the $\langle\cdot\rangle^{I}_{\trianglelefteq p}$ operator. It is possible to add the time-bounded reachability as in CSL by means of transient probability distribution Eq. (6), but without any nesting. Therefore, future work will consist of investigating time-bounded reachability as well as long-run operators for ICTMCs.

References

1. Aho, A. V., Hopcroft, J. E., Ullman, J. D.: *Design and Analysis of Computer Algorithms*. Addison-Wesley, 1974.
2. Alur, R., Courcoubetis, C., Dill, D. L.: Model-checking for real-time systems. *Proceedings of the Fifth Annual IEEE Symposium on Logic in Computer Science*, pages 414–425, 1990.
3. Alur, R., Dill, D. L.: A theory of timed automata. *Theoretical Computer Science*, **126**(2): 183–235, 1994.
4. Aziz, A., Sanwal, K.,Singhal, V., Brayton, R.: Model checking continuous time Markov chains. *ACM Trans. on Comp. Logic*, **1**(1): 162–170, 2000.
5. Baier, C., Haverkort, B. R., Hermanns, H., Katoen, J.-P.: Model-checking algorithms for continuous-time Markov chains. *IEEE Trans. on Softw. Eng.*, **29**(6): 524–541, 2003.
6. Buchholz, P.: Exact and ordinary lumpability in finite Markov chains. *J. of Applied Probability*, **31**: 59–75, 1994.
7. Clark, G., Gilmore, S., Hillston, J., Ribaudo, M.: Exploiting modal logic to express performance measures. *Computer Performance Evaluation: Modeling Techniques and Tools*, LNCS 1786: 247–261, Springer-Verlag, 2000.
8. Desharnais, J., Panangaden, P: Continuous stochastic logic characterizes bisimulation of continuous-time Markov processes. *J. Log. Algebr. Program.*, **56**(1-2): 99-115, 2003.
9. Hamming, R. W.: *Numerical Methods for Scientists and Engineers*. McGraw-Hill, 1973.
10. Han, T., Katoen, J.-P., Mereacre, A.: Compositional modeling and minimization of time-inhomogeneous Markov chains. *Hybrid Systems: Computation and Control*, LNCS 4981: 244-258, Springer-Verlag, 2008.
11. Hennessy, M., Milner, R.: Algebraic laws for nondeterminism and concurrency. *J. ACM*, **32**(1): 137–161, 1985.
12. Henrici, P., Kenan, W. R.: *Applied & Computational Complex Analysis: Power Series Integration Conformal Mapping Location of Zero*. John Wiley & Sons, 1988.
13. Hermanns, H.: *Interactive Markov Chains: The Quest for Quantified Quality*. LNCS 2428, Springer, 2002.
14. Hillston, J.: *A Compositional Approach to Performance Modeling*. Cambridge University Press, 1996.
15. Katoen, J.-P., Khattri, M., Zapreev, I.S.: A Markov reward model checker. *Quantitative Evaluation of Systems (QEST)*, IEEE CS Press, pp. 243–245, 2005.
16. Kwiatkowska, M. Z., Norman, G., Parker, D.A.: Probabilistic symbolic model checking using PRISM: a hybrid approach. *J. on Software Tools for Technology Transfer*, **6**(2): 128–142, 2004.
17. Laguerre, E.: Sur la théorie des équations numériques. *J. Math. Pures Appl. (3e série)*, **9**: 99–146, 1883.
18. Larsen, K. G., Skou, A.: Bisimulation through probabilistic testing. *Inf. Comput.*, **94**(1): 1–28, 1991.
19. Parma, A., Segala, R.: Logical characterizations of bisimulations for discrete probabilistic systems. *FoSSaCS*, LNCS 4423: 287–301, 2007.
20. Reif, J. H.: An $\mathcal{O}\left(n \log^3 n\right)$ algorithm for the real root and symmetric tridiagonal eigenvalue problems. *34th Annual IEEE Conference on Foundations of Computer Science (FOCS '93)*, pp. 626-635, 1993.
21. Segala, R.: *Modeling and Verification of Randomized Distributed Real-Time Systems*. PhD thesis, MIT, Dept. of Electrical Eng. and Computer Sci., 1995.