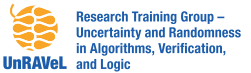
Reasoning about Expected Runtimes of Probabilistic Programs

(and Quantitative Separation Logic, time permitting)

Benjamin Lucien Kaminski



Dagstuhl Seminar 19371: **Deduction Beyond Satisfiability**

September 11, 2019, Schloss Dagstuhl, Germany

Probabilistic Program Applications

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symmetry breaking

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symmetry breaking

speed–up

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cryptography and security

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machine learning & AI

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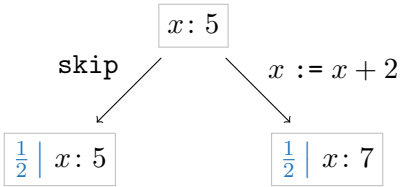

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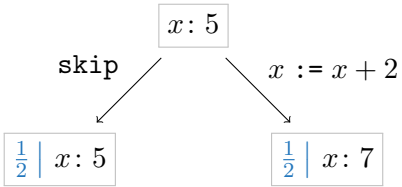

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- ERT of C can be infinite, even if C terminates almost–surely[1]

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Weakest Precondition Reasoning for Expected Runtimes

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- Set of runtimes $\mathbb{T} \;=\; \left\{ t \;\middle|\; t\colon \text{ States} \to \mathbb{R}^{\infty}_{\geq 0} \right\}$, a complete lattice with partial order

$$r \preceq t \qquad \text{iff} \qquad \forall\, \sigma \in \text{States}\colon \quad r(\sigma) \;\leq\; t(\sigma)$$

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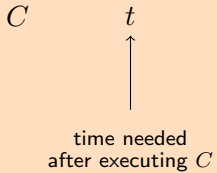

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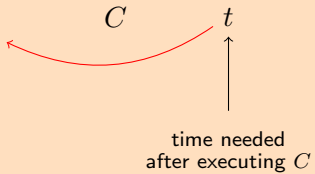

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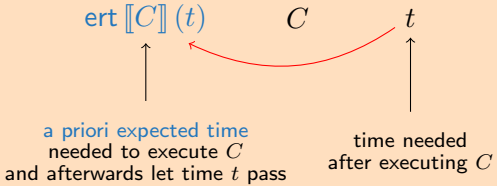

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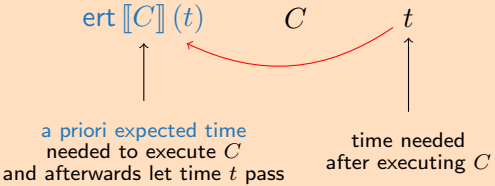

ERT in Terms of ert

$$\mathsf{ert}[\![C]\!](\mathbf{0})(\sigma) \;=\; \text{“ERT of } C \text{ on input } \sigma\text{”}$$

ert of Skipping

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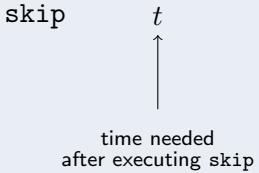

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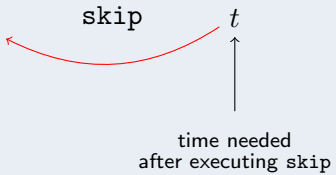

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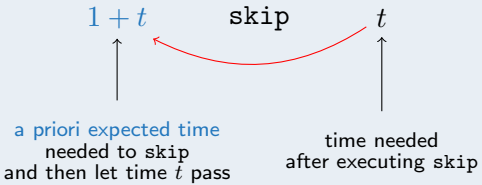

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$x := z + 1$ $\qquad$ x^2

time needed
after executing $x := z + 1$

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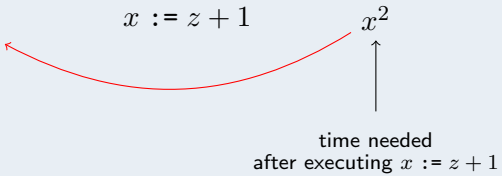

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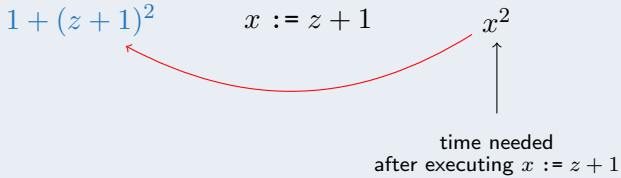

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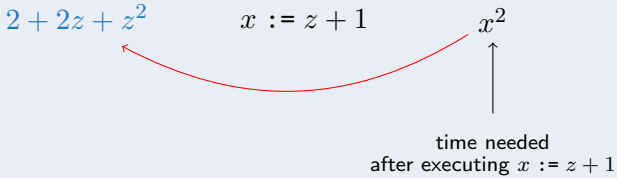

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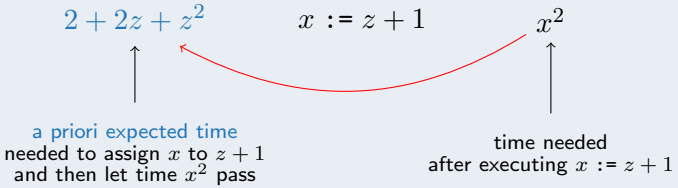

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$\texttt{while}\,(\varphi)\,\{C'\}$	$\mathsf{lfp}\,X \bullet \quad 1 + [\neg\varphi] \cdot t + [\varphi] \cdot \mathsf{ert}\,[\![C']\!]\,(X)$

The ert Transformer in Action

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$$/\!/\!/ \; \mathbf{24}$$

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Induction for Loops

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- Φ monotonic implies lfp exists [Knaster'28, Tarski'49, Kleene'62, *Folklore*]
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- Reasoning about lfp needed: Induction! [Park '69]

$$\Phi(I) \preceq I \quad \text{implies} \quad \text{lfp}\,\Phi \preceq I$$

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$$\texttt{while}\,(x > 0)$$
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Postruntime: 1 (expected time of executing the loop and then needing 1 time unit)

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Candidate for upper bound: $\boldsymbol{I \;=\; 2 + [x > 0]\,6x}$

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Thus: $\boldsymbol{\mathsf{ert}\,[\![\texttt{while}\ldots]\!]\,(1) \quad \preceq \quad 2+[x>0]\,6x}$

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- **Verifying lower bounds: Does $I \preceq \mathrm{lfp}\ \Phi$ hold?**

Quantitative Separation Logic

Quantitative Separating Conjunction [POPL'19]

Classical separating conjunction:

$$(s,h) \models F \star G \qquad \text{iff}$$

$$\exists\, h_1, h_2 \colon \quad h = h_1 \star h_2 \quad \text{and} \quad (s,h_1) \models F \quad \text{and} \quad (s,h_2) \models G$$

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Naive quantitative separating conjunction:

$$\big(f \star g\big)(s,\, h) \quad = \quad \exists\, h_1, h_2 \colon \; [h = h_1 \star h_2] \cdot f(s,h_1) \cdot g(s,h_2)$$

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Meaningful quantitative separating conjunction:

$$f \star g \;=\; \lambda(s,h).\, \max_{h_1,h_2} \Big\{\, f(s,h_1) \cdot g(s,h_2) \;\Big|\; h = h_1 \star h_2 \,\Big\}$$

Quantitative Separating Implication [POPL'19]

Classical separating implication:

$$(s,h) \models F \mathrel{-\!\!\star} G \quad \text{iff} \quad \forall\, h' \text{ with } h' \perp h \wedge (s,h') \models F\colon \quad (s, h \star h') \models G$$

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Quantitative separating implication:

$$f \mathrel{-\!\!\!\star} g \quad = \quad \lambda(s,h).\ \inf_{h'} \left\{ \frac{g(s,h\star h')}{f(s,h')} \;\middle|\; \begin{array}{l} h' \perp h \text{ and } f(s,h') > 0 \text{ and} \\ \underline{\text{not}}\ f(s,h') = \infty = g(s,h\star h') \end{array} \right\}$$

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Notice:

$$[F] \mathrel{-\!\!*} g \quad = \quad \lambda(s,h).\ \inf_{h'} \left\{ g(s,h\star h') \;\middle|\; h' \perp h \text{ and } (s,h') \models F \right\}$$

Modus Ponens

Classical modus ponens:

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Modus ponens of quantitative separation logic:

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Adjointness of $\mathrel{-\!\!*}$ and $\star$

Classical adjointness:

$$\big((F \star G) \implies J\big) \quad \iff \quad \big(F \implies (G \mathrel{-\!\!*} J)\big)$$

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My personal Aha Erlebnis:

$$a - b \;\preceq\; c \qquad \text{iff} \qquad a \;\preceq\; b + c$$

$$a \cdot b \;\preceq\; c \qquad \text{iff} \qquad a \;\preceq\; c \,/\, b$$

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