A Weakest Pre–Expectation Semantics for Mixed–Sign Expectations

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Example of a Probabilistic Program

$$
\begin{aligned}
&\{c \texttt{ := } 0\}\ [1/2]\ \{c \texttt{ := } 1\};\\
&\texttt{if}\ (c = 1)\ \{x \texttt{ := } 1\}\ \texttt{else}\ \{x \texttt{ := } 2x + 1\};\\
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\end{aligned}
$$

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Formal verification of probabilistic programs!

Classical Weakest Pre–Expectations

The Non–Negative Case

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post–expectation f
evaluated in final states
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$$\lambda\sigma\bullet\ \mathsf{EV}_{[\![C]\!]_\sigma}(f) \stackrel{!}{=} \mathsf{wp}\,[C]\,(f) \qquad C \qquad f$$

weakest pre–expectation
of C with respect to f
evaluated in initial states
before executing C

post–expectation f
evaluated in final states
after termination of C

Weakest Pre–Expectation Reasoning

Example of wp Reasoning

$\{c \texttt{ := } 0\}\ [1/2]\ \{c \texttt{ := } 1\}\,;$

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x

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$$\qquad x$$

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Weakest Pre–Expectation Reasoning

Example of wp Reasoning

$\{c \texttt{ := } 0\}\ [1/2]\ \{c \texttt{ := } 1\}\,;$

$\quad [\![c = 1]\!] \cdot 1 + [\![c = 0]\!] \cdot (2x + 1)$

$\texttt{if}\ (c = 1)\ \{x \texttt{ := } 1\}\ \texttt{else}\ \{x \texttt{ := } 2x + 1\};$

$\quad x$

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$$\mathsf{wp}\,[\{\ldots\}\;[1/2]\;\{\ldots\}]\,(1 + [\![c = 0]\!] \cdot 2x)$$

$\{c \mathrel{:=} 0\}\;[1/2]\;\{c \mathrel{:=} 1\}\,;$

$$1 + [\![c = 0]\!] \cdot 2x$$

$\texttt{if}\;(c = 1)\;\{x \mathrel{:=} 1\}\;\texttt{else}\;\{x \mathrel{:=} 2x + 1\};$

$$x$$

`skip`

$$x$$

Weakest Pre–Expectation Reasoning

Example of wp Reasoning

$$\frac{1}{2} \cdot (1 + [\![0 = 0]\!] \cdot 2x) + \frac{1}{2} \cdot (1 + [\![1 = 0]\!] \cdot 2x)$$

$\{c \texttt{ := } 0\}\ [1/2]\ \{c \texttt{ := } 1\}\,;$

$$1 + [\![c = 0]\!] \cdot 2x$$

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$$1 + x$$

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$$1 + [\![c = 0]\!] \cdot 2x$$

`if` $(c = 1)\ \{x := 1\}$ `else` $\{x := 2x + 1\};$

$$x$$

`skip`

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The wp Transformer for While Loops

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Complete partial order on expectations:

$$f_1 \;\preceq\; f_2 \quad \text{iff} \quad \forall \sigma\colon f_1(\sigma) \;\leq\; f_2(\sigma)$$

The Motivation

Example of wp Reasoning

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Neither post–expectation x nor any of the pre–expectations are p<u>ro</u>p<u>er</u> expectations!

Mixed–Sign Weakest Pre–Expectations

The Non–Non–Negative Case

Our Solution: Integrability–Witnessing Expectations

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- Integrability–witnessing pairs:

$$(f,\, g)$$

such that $f \in \mathbb{E}^\star$, $g \in \mathbb{E}$, and $|f| \leq g$

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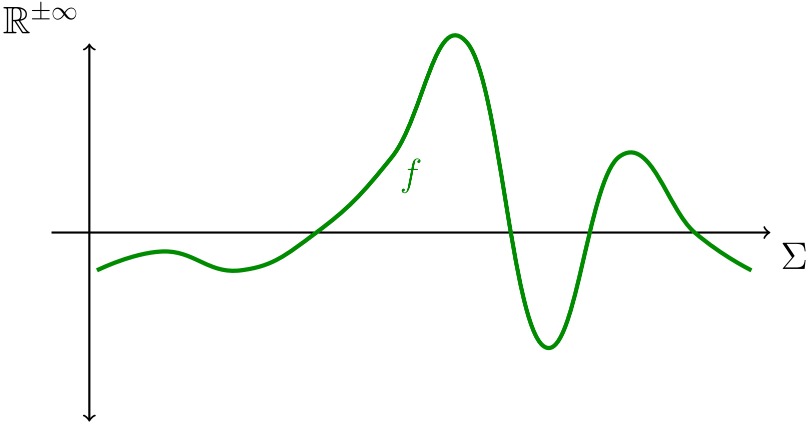

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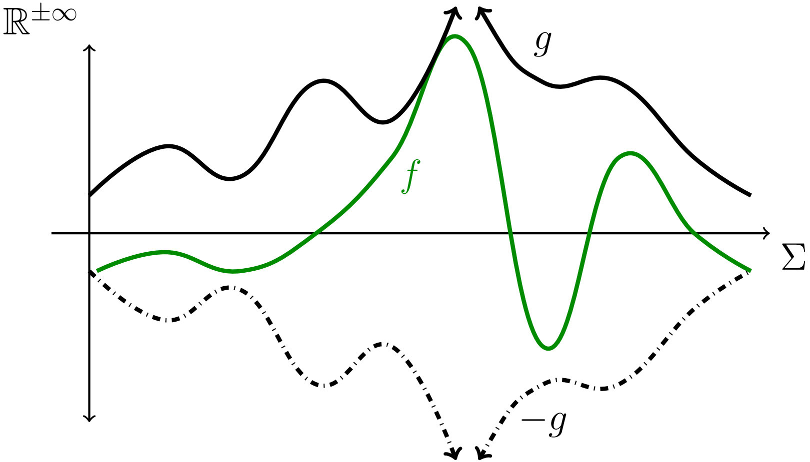

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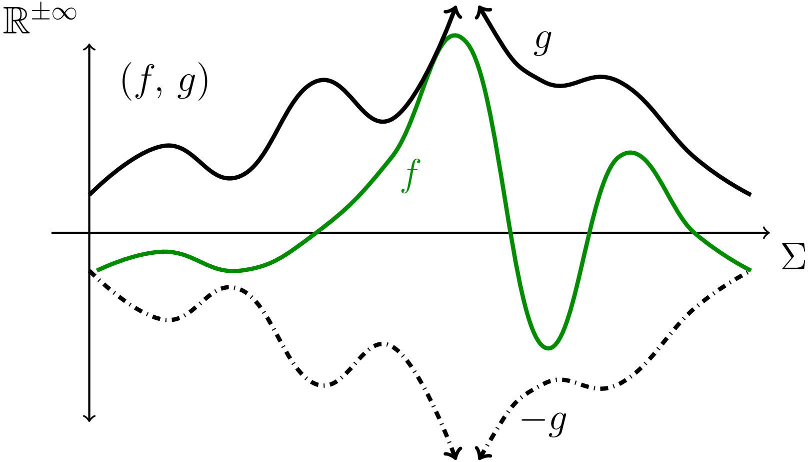

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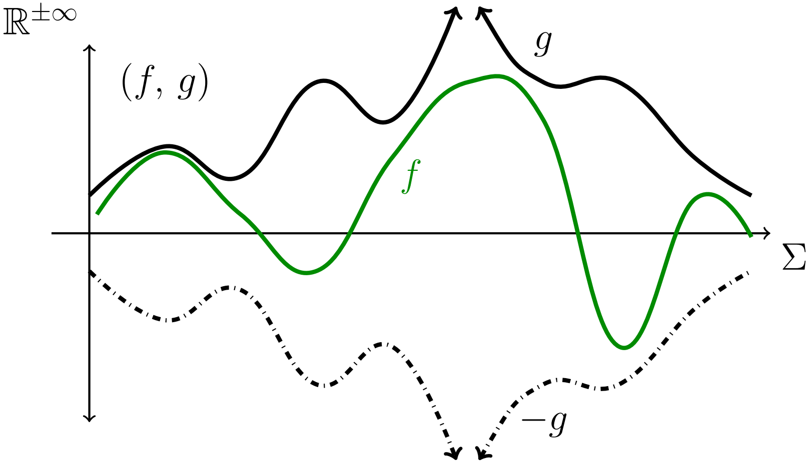

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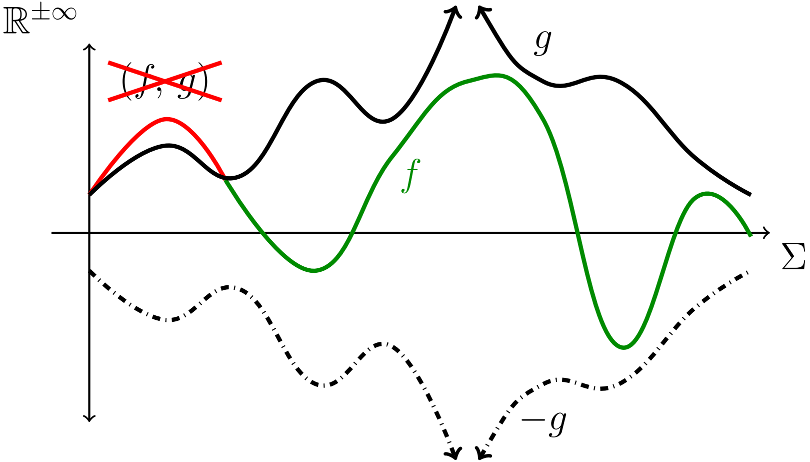

wp with Integrability–Witnessing Pairs

Example of Integrability–Witnessing Pair Reasoning

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- Consider `while` $(\xi)\ \{C\}$
- Recall wp $[\texttt{while}\ (\xi)\ \{C\}]\,(f) = \sup_n F_f^n(0)$ (KFPT)
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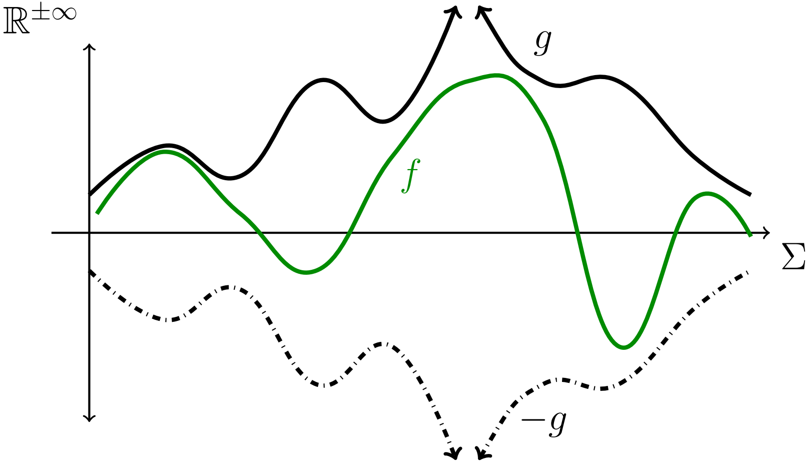

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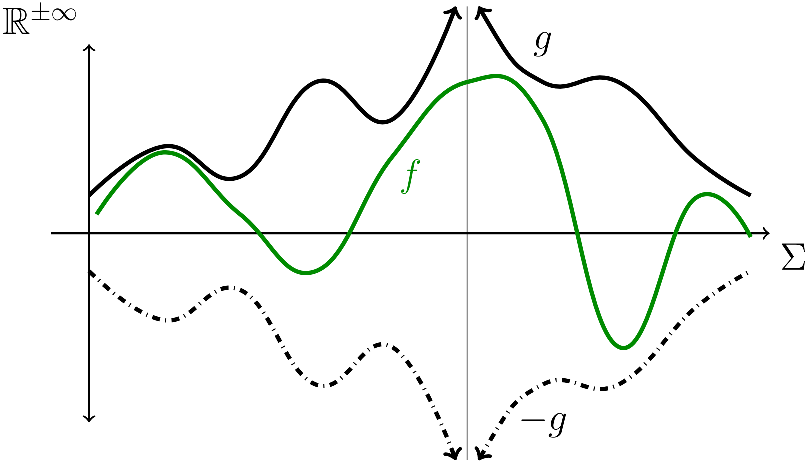

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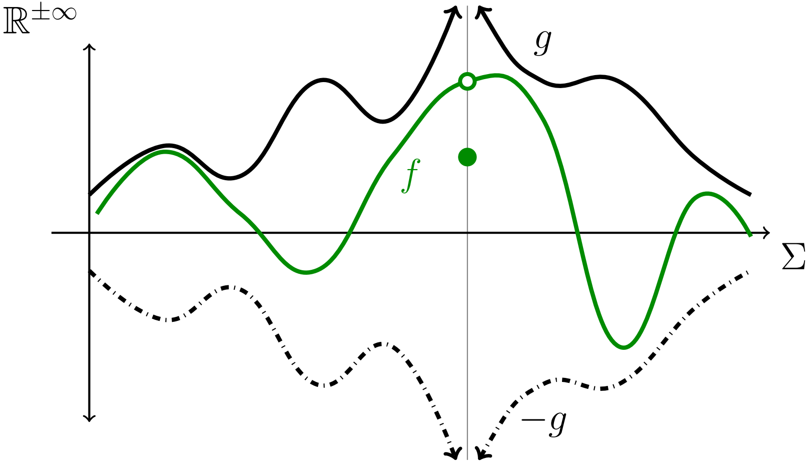

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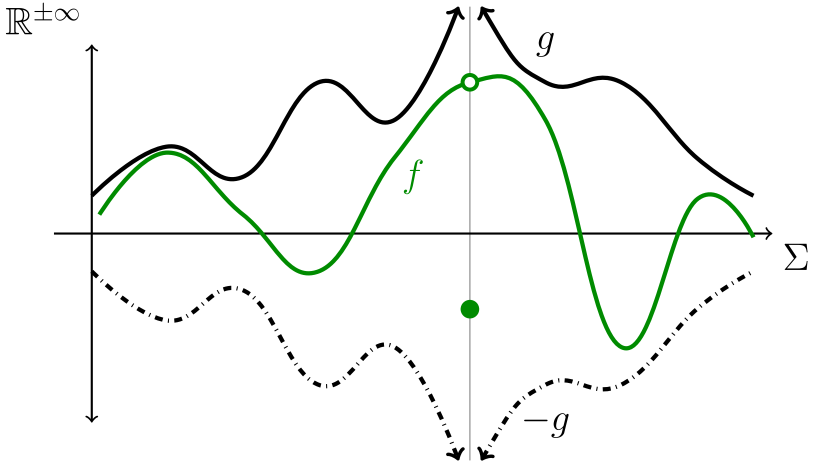

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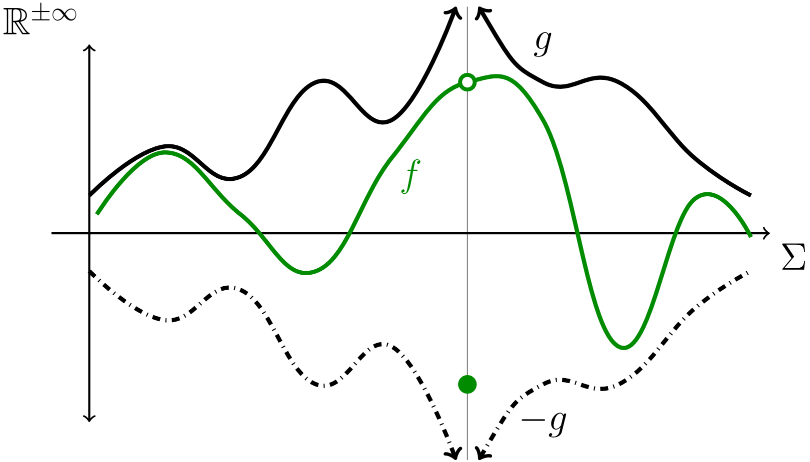

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- Still we can define a wp transformer acting on IWEs

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Invariant Rule for While–Loops

Let $I, G \in \mathbb{E}$ and $\{H_n\}_{n \in \mathbb{N}} \subseteq \mathbb{E}$. If

$$F_{|f|+f}(I) \leq I, \quad F_g(G) \leq G,$$
$$H_0 \leq F_{|f|}(0), \quad \text{and} \quad H_{n+1} \leq F_{|f|}(H_n),$$

then

$$\mathsf{wp}[\mathtt{while}\,(\xi)\,\{C'\}]\wr f,\, g \wr \;\sqsubseteq\; \Big\wr\, I - \sup_n H_n,\; 2 \cdot G \,\Big\wr\,.$$

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Thank you for your kind attention!

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- If $\sum a_i$ abs. conv., then $\sum a_i = \sum \big(|a_i| + a_i\big) - \sum |a_i|$
- Apply this principle to $F_{\wr f,\, g \wr}$:

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Invariant Rule for While–Loops

Let $I, G \in \mathbb{E}$ and $\{H_n\}_{n \in \mathbb{N}} \subseteq \mathbb{E}$. If

$$F_{|f|+f}(I) \leq I, \quad F_g(G) \leq G,$$
$$H_0 \leq F_{|f|}(0), \quad \text{and} \quad H_{n+1} \leq F_{|f|}(H_n),$$

then

$$\mathtt{wp}[\mathtt{while}\,(\xi)\,\{C'\}]\wr f,\, g \int \;\sqsubseteq\; \Big\wr I - \sup_n H_n,\; 2 \cdot G \Big\int .$$

Backup Slides: Upper Bounds for wp of While–Loops

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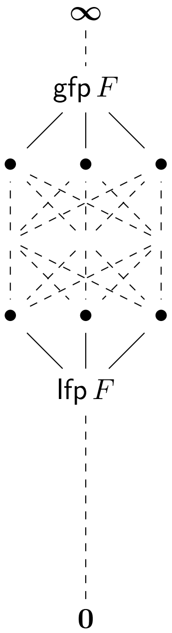
Backup Slides: Park's Lemma



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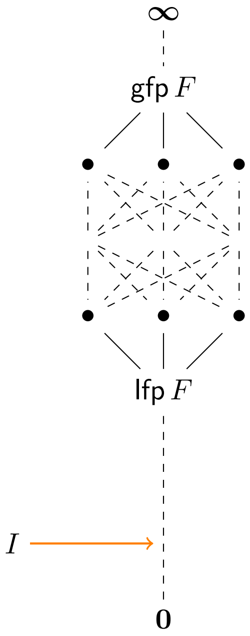

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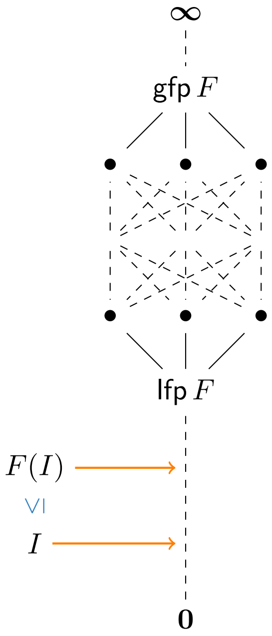

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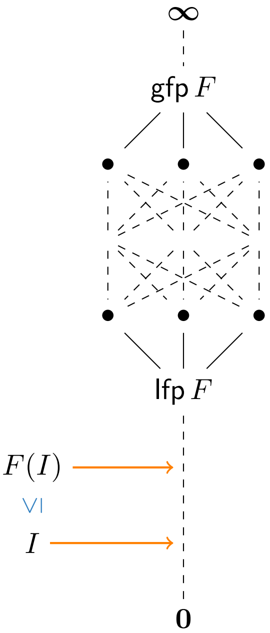

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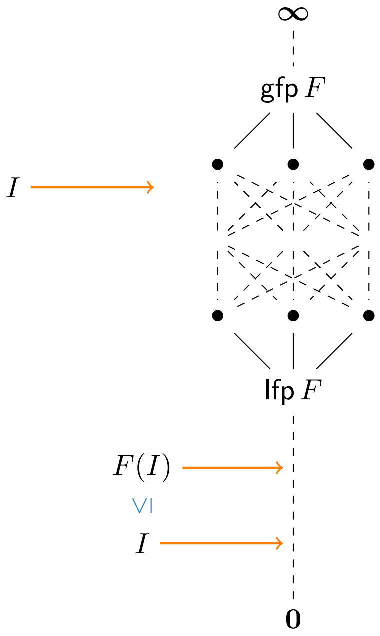

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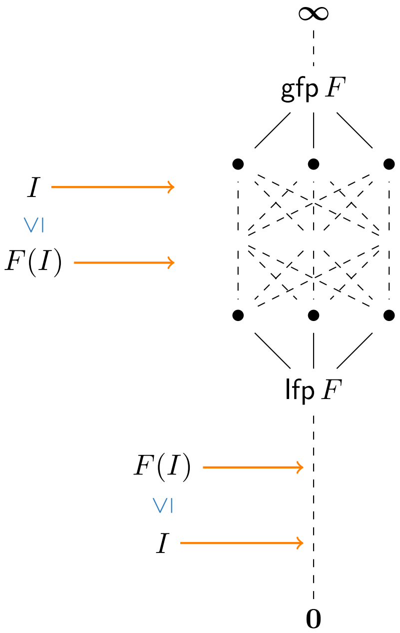

PPDL's While Rule [Kozen '85, p. 168]

$$\neg B \cdot f + B \cdot \langle\, C \,\rangle\, I \;\preceq\; I \quad \text{implies} \quad \langle\, \texttt{while}\,(B)\,\{C\}\,\rangle f \;\preceq\; I\;,$$

for $f \succeq 0$.