The Satisfiability of Some Simple Probabilistic Logics

Souymodip Chakraborty and **Joost-Pieter Katoen**

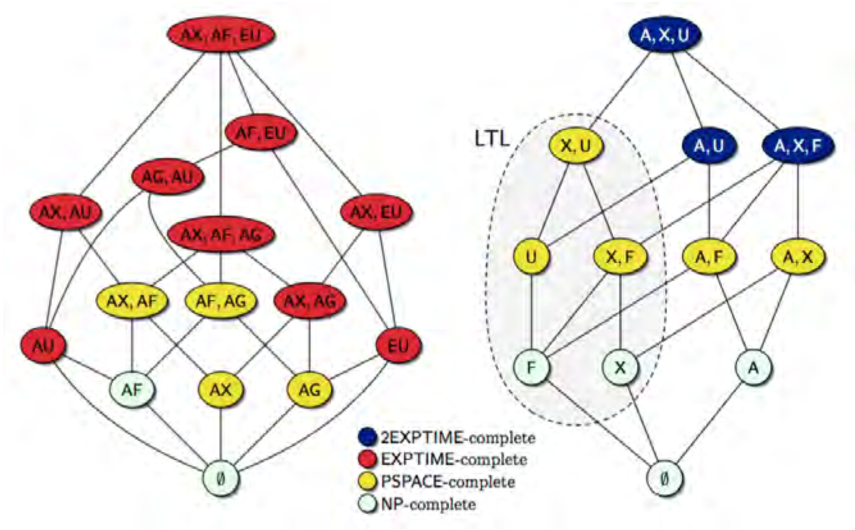
RWTH Aachen University
Software Modeling and Verification Group

LICS 2016, New York, USA

July 5, 2016

CTL and CTL* satisfiability

[Meier *et al.*, 2008]



Satisfiability for probabilistic CTL

Satisfiability of probabilistic logics such as PCTL (and fragments thereof) is largely open.

In particular for quantitative fragments.[1]

[1] Notable exception: Kozen's probabilistic PDL.

Probabilistic CTL

[Hansson & Jonsson, 1989]

Probabilistic CTL

[Hansson & Jonsson, 1989]

- PCTL interpretation is Boolean, i.e., a formula is satisfied or not.

Probabilistic CTL

[Hansson & Jonsson, 1989]

- PCTL interpretation is Boolean, i.e., a formula is satisfied or not.
- For path-formula φ and threshold $\succ p$ with $\succ \in \{ >, \geqslant \}$ and $p \in \mathbb{Q}$:

  PCTL-formula $[\varphi]_{\succ p}$ denotes

  all paths satisfying φ occur with probability $\succ p$
- $[\cdot]_{\succ p}$ is probabilistic counterpart of CTL path-quantifiers $\exists$ and $\forall$.

Probabilistic CTL

[Hansson & Jonsson, 1989]

- PCTL interpretation is Boolean, i.e., a formula is satisfied or not.
- For path-formula φ and threshold $\succ p$ with $\succ \in \{ >, \geqslant \}$ and $p \in \mathbb{Q}$:

  PCTL-formula $[\varphi]_{\succ p}$ denotes

  all paths satisfying φ occur with probability $\succ p$
- $[\cdot]_{\succ p}$ is probabilistic counterpart of CTL path-quantifiers $\exists$ and $\forall$.
- Examples: $[\Diamond a]_{>1/2}$ and $[\Box(\neg a \wedge [\Diamond a]_{>0})]_{>0}$.

Probabilistic CTL

[Hansson & Jonsson, 1989]

- PCTL interpretation is Boolean, i.e., a formula is satisfied or not.
- For path-formula φ and threshold $\succ p$ with $\succ \in \{ >, \geqslant \}$ and $p \in \mathbb{Q}$:

  PCTL-formula $[\varphi]_{\succ p}$ denotes

  all paths satisfying φ occur with probability $\succ p$
- $[\cdot]_{\succ p}$ is probabilistic counterpart of CTL path-quantifiers $\exists$ and $\forall$.
- Examples: $[\Diamond\, a]_{>1/2}$ and $[\Box(\neg a \wedge [\Diamond\, a]_{>0})]_{>0}$.

PCTL model checking is in P.

PCTL satisfiability is a long-standing open problem.

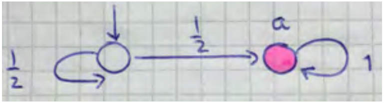
Zero probabilities

[2] On finite Markov chains.

Zero probabilities

Almost sure $\neq$ always:

$[\Diamond a]_{=1} \not\equiv \forall\Diamond a$

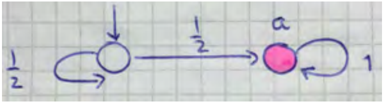


[2] On finite Markov chains.

Zero probabilities

Almost sure $\neq$ always:

$[\Diamond a]_{=1} \not\equiv \forall\Diamond a$



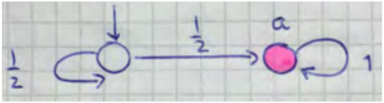
CTL versus qualitative PCTL – only thresholds >0 and $=1$ [BK08]

[2] On finite Markov chains.

Zero probabilities

Almost sure $\neq$ always:

$[\Diamond a]_{=1} \not\equiv \forall \Diamond a$



CTL versus qualitative PCTL – only thresholds >0 and $=1$ **[BK08]**

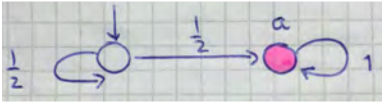
1. There is no CTL formula that is equivalent to $[\Diamond a]_{=1}$.

[2] On finite Markov chains.

Zero probabilities

Almost sure $\neq$ always:

$[\Diamond a]_{=1} \;\; \not\equiv \forall\Diamond a$



CTL versus qualitative PCTL – only thresholds >0 and $=1$ **[BK08]**

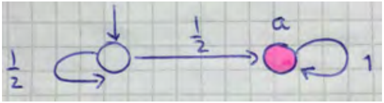
1. There is no CTL formula that is equivalent to $[\Diamond a]_{=1}$.
2. There is no PCTL formula that is equivalent to $\forall\Diamond a$.

[2] On finite Markov chains.

Zero probabilities

Almost sure $\neq$ always:

$[\Diamond a]_{=1} \not\equiv \forall\Diamond a$



CTL versus qualitative PCTL – only thresholds >0 and $=1$ **[BK08]**

1. There is no CTL formula that is equivalent to $[\Diamond a]_{=1}$.
2. There is no PCTL formula that is equivalent to $\forall\Diamond a$.
3. Persistency like $[\Diamond\Box\, a]_{>0}$ is expressible in PCTL[2] but not in CTL.

[2] On finite Markov chains.

Qualitative PCTL

[Brazdil *et al.*, 2008]

Qualitative PCTL

[Brazdil *et al.*, 2008]

$\Phi \;=\; [\Box\,(\neg a \wedge [\Diamond\, a]_{>0})]_{>0}$ has no finite model.

Qualitative PCTL

[Brazdil *et al.*, 2008]

$\Phi \;=\; [\Box\,(\neg a \wedge [\Diamond\, a]_{>0})]_{>0}$ has no finite model.

- For every finite Markov chain, $[\Diamond\, a]_{>0}$ implies $[\Diamond\, a]_{>\epsilon}$ for some $\epsilon > 0$.

$\Rightarrow$ The probability of all runs satisfying $\Box\,(\neg a \wedge [\Diamond\, a]_{>0})$ is zero.

- . . . . . . but Φ has an infinite model:

Qualitative PCTL

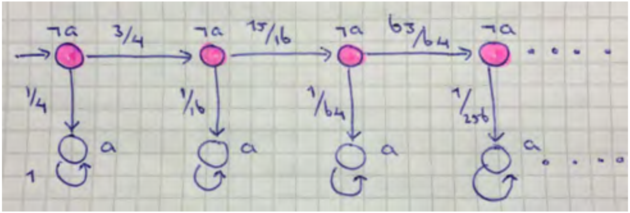
[Brazdil *et al.*, 2008]

$\Phi \;=\; [\Box\,(\neg a \wedge [\Diamond\, a]_{>0})]_{>0}$ has no finite model.

- For every finite Markov chain, $[\Diamond\, a]_{>0}$ implies $[\Diamond\, a]_{>\epsilon}$ for some $\epsilon > 0$.

$\Rightarrow$ The probability of all runs satisfying $\Box\,(\neg a \wedge [\Diamond\, a]_{>0})$ is zero.

- . . . . . . but Φ has an infinite model:



The only run satisfying $\Box\,(\neg a \wedge [\Diamond\, a]_{>0})$ has positive probability.

Qualitative PCTL

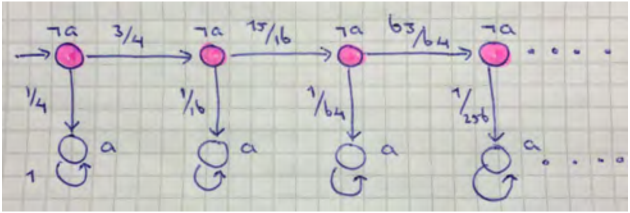
[Brazdil *et al.*, 2008]

$$\Phi \; = \; [\Box\,(\neg a \wedge [\Diamond\, a]_{>0})]_{>0} \text{ has no finite model.}$$

- For every finite Markov chain, $[\Diamond\, a]_{>0}$ implies $[\Diamond\, a]_{>\epsilon}$ for some $\epsilon > 0$.

$\Rightarrow$ The probability of all runs satisfying $\Box\,(\neg a \wedge [\Diamond\, a]_{>0})$ is zero.

- ...... but Φ has an infinite model:



The only run satisfying $\Box\,(\neg a \wedge [\Diamond\, a]_{>0})$ has positive probability.

Qualitative PCTL has – in contrast to CTL – no finite model property.

Rational models

[Bertrand *et al.*, 2012]

PCTL formulas may have only models with irrational probabilities if model size is bounded.

Rational models

[Bertrand *et al.*, 2012]

PCTL formulas may have only models with irrational probabilities if model size is bounded.

$$\Phi \;=\; 1 \wedge [\Box\,(\mathsf{sink} \wedge \mathsf{topo} \wedge \mathsf{prob})]_{=1}$$

where

- $\mathsf{sink} \;=\; 0 \rightarrow [\bigcirc\, 0]_{=1}$
- $\mathsf{topo} \;=\; (1 \rightarrow [\bigcirc\,(2 \vee 0)]_{=1}) \wedge (2 \rightarrow [\bigcirc\,(3 \vee 0)]_{=1}) \wedge (3 \rightarrow [\bigcirc\,(1 \vee 0)]_{=1})$
- $\mathsf{prob} \;=\; \left(1 \rightarrow [(1 \vee 2)\,\mathsf{U}\,3]_{=1/2}\right) \wedge \left(2 \rightarrow [(2 \vee 3)\,\mathsf{U}\,1]_{=1/2}\right) \wedge \left(3 \rightarrow [(3 \vee 1)\,\mathsf{U}\,2]_{=1/2}\right)$

Rational models

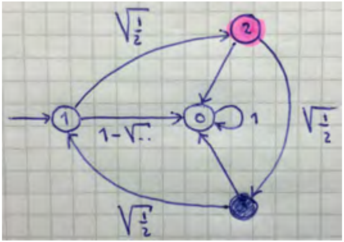
[Bertrand *et al.*, 2012]

PCTL formulas may have only models with irrational probabilities if model size is bounded.

$$\Phi \;=\; 1 \wedge [\Box\,(\mathsf{sink} \wedge \mathsf{topo} \wedge \mathsf{prob})]_{=1}$$

where

- $\mathsf{sink} \;=\; 0 \rightarrow [\bigcirc\, 0]_{=1}$
- $\mathsf{topo} \;=\; (1 \rightarrow \forall \bigcirc\, (2 \vee 0)) \wedge (2 \rightarrow \forall \bigcirc\, (3 \vee 0)) \wedge (3 \rightarrow \forall \bigcirc\, (1 \vee 0))$
- $\mathsf{prob} \;=\; \left(1 \rightarrow [(1 \vee 2)\, \mathsf{U}\, 3]_{=1/2}\right) \wedge \left(2 \rightarrow [(2 \vee 3)\, \mathsf{U}\, 1]_{=1/2}\right) \wedge \left(3 \rightarrow [(3 \vee 1)\, \mathsf{U}\, 2]_{=1/2}\right)$



Only four-state model for Φ.

Rational models

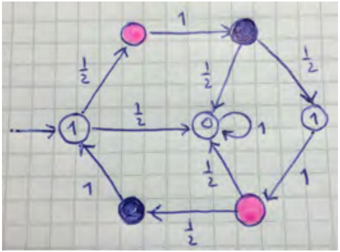
[Bertrand *et al.*, 2012]

PCTL formulas may have only models with irrational probabilities if model size is bounded.

$$\Phi \;=\; 1 \wedge [\Box\,(\mathsf{sink} \wedge \mathsf{topo} \wedge \mathsf{prob})]_{=1}$$

where

- $\mathsf{sink} \;=\; 0 \rightarrow [\bigcirc\, 0]_{=1}$
- $\mathsf{topo} \;=\; (1 \rightarrow \forall \bigcirc\, (2 \vee 0)) \wedge (2 \rightarrow \forall \bigcirc\, (3 \vee 0)) \wedge (3 \rightarrow \forall \bigcirc\, (1 \vee 0))$
- $\mathsf{prob} \;=\; \big(1 \rightarrow [(1 \vee 2)\, \mathsf{U}\, 3]_{=1/2}\big) \wedge \big(2 \rightarrow [(2 \vee 3)\, \mathsf{U}\, 1]_{=1/2}\big) \wedge \big(3 \rightarrow [(3 \vee 1)\, \mathsf{U}\, 2]_{=1/2}\big)$



A rational model for Φ.

What is known about PCTL satisfiability?

[3] using linear arithmetic and uninterpreted functions, which is NP-complete too.

What is known about PCTL satisfiability?

Known facts:

1. Qualitative PCTL: decidable, no finite model property. [HS86]
2. (Finite) satisfiability of qualitative PCTL is EXPTIME-c. [BFKK08]

[3] using linear arithmetic and uninterpreted functions, which is NP-complete too.

What is known about PCTL satisfiability?

Known facts:

1. Qualitative PCTL: decidable, no finite model property. [HS86]
2. (Finite) satisfiability of qualitative PCTL is EXPTIME-c. [BFKK08]
3. Satisfiable PCTL-formulas have model of branching degree $\leqslant |\Phi|+2$.

[3] using linear arithmetic and uninterpreted functions, which is NP-complete too.

What is known about PCTL satisfiability?

Known facts:

1. Qualitative PCTL: decidable, no finite model property. [HS86]
2. (Finite) satisfiability of qualitative PCTL is EXPTIME-c. [BFKK08]
3. Satisfiable PCTL-formulas have model of branching degree $\leqslant |\Phi|+2$.
4. Bounded satisfiability: for PCTL formula Φ and bound $k \in \mathbb{N}$:

   does there exist a rational Markov chain with $|S| \leqslant k$ satisfying Φ?

   is NP-complete
5. and can be reduced to SMT checking[3] in $\mathcal{O}(|\text{sub}(\Phi)|)$ time. [BFS12]

[3] using linear arithmetic and uninterpreted functions, which is NP-complete too.

What is new about PCTL satisfiability?

Our results:

1. Bounded PCTL-formulas have a finite model, exponential in size Φ.
2. Satisfiability of PCTL with only nested $\bigcirc$-modalities is PSPACE-c.
3. Satisfiability of bounded PCTL is in NEXPTIME (in size of Φ)
4. . . . . . . and is EXPTIME-hard (in encoding of Φ).

What is new about PCTL satisfiability?

Our results:

1. Bounded PCTL-formulas have a finite model, exponential in size Φ.
   - Path-formulas are restricted to $\varphi ::= \bigcirc \Phi \;\Big|\; \Phi_1 \, \mathsf{U}^{n} \, \Phi_2$
   - Examples: $[\Diamond^4 a]_{>1/2}$ and $\left[\Diamond^3 [\Box^{10} a]_{=1}\right]_{>1/3}$
   - A formula has a tree model of depth ord(Φ) and degree $|\text{sub}(\Phi)+1|$.
2. Satisfiability of PCTL with only nested $\bigcirc$-modalities is PSPACE-c.
3. Satisfiability of bounded PCTL is in NEXPTIME (in size of Φ)
4. . . . . . . and is EXPTIME-hard (in encoding of Φ).

What is new about PCTL satisfiability?

Our results:

1. Bounded PCTL-formulas have a finite model, exponential in size Φ.
2. Satisfiability of PCTL with only nested $\bigcirc$-modalities is PSPACE-c.
   - n-th level formulas: $[\bigcirc \Phi]_{\succ p}$ with Φ of level $n-1$
   - $\in$ PSPACE: satisfiability by NTM with oracle Ω_{n-1} in poly-time selecting weighted cover for $[\bigcirc f]_{\succ p}$-formulas
   - PSPACE-hardness: by satisfiability of K-logic
3. Satisfiability of bounded PCTL is in NEXPTIME (in size of Φ)
4. ...... and is EXPTIME-hard (in encoding of Φ).

What is new about PCTL satisfiability?

Our results:

1. Bounded PCTL-formulas have a finite model, exponential in size Φ.
2. Satisfiability of PCTL with only nested $\bigcirc$-modalities is PSPACE-c.
3. Satisfiability of bounded PCTL is in NEXPTIME (in size of Φ)
   - non-deterministically guess a tree model exponential in $|\Phi|$
   - NP-algorithm for labelling this tree model
   - this yields real, non-linear in-equations, polynomial in input size
   - checking inequations by using first-order theory of reals.
4. ...... and is EXPTIME-hard (in encoding of Φ).
   - reduction from acceptance problem for alternating poly-space TM.

Summary

Logic	Decidable?	Finite model?	Complexity?
CTL	$\surd$	$\mathcal{O}\left(2^{\lvert\Phi\rvert}\right)$	EXPTIME-c.
CTL*	$\surd$	$\mathcal{O}\left(2^{\lvert\Phi\rvert}\right)$	2EXPTIME-c.
qualitative PCTL	$\surd$	no	EXPTIME-c.
PX_{ω}	$\surd$ (tree)	$\mathcal{O}(\lvert\Phi\rvert)$	PSPACE-c.
bounded PCTL	$\surd$ (tree)	$\mathcal{O}\left(\text{ord}(\Phi) + \lvert\text{sub}(\Phi)\rvert\right)$	NEXPTIME EXPTIME-hard
PCTL	open	no	open

"there are structural regularities in PCTL models which suggest that the [satisfiability] problem might be decidable"

Propositional μ-calculus

The FMP for CTL, CTL* etc. follows from the FMP of $L\mu$.

[K83,K88,SE84]

What about satisfiability and FMP results for probabilistic μ-calculus?

The probabilistic μ-calculi zoo

The probabilistic μ-calculi zoo

- qmμ = quantitative interpretation of μ-calculus [HK97,MM97]
  - $\cup$ = max, $\cap$ = min, no PCTL, game characterization on finite models

The probabilistic μ-calculi zoo

- qmμ = quantitative interpretation of μ-calculus [HK97,MM97]
  - $\cup$ = max, $\cap$ = min, no PCTL, game characterization on finite models
- GPL = extension with finite nesting of $[\cdot]_{\succ p}$ quantifications [CPN99]
  - expresses PCTL* but neither $\exists\Box a$ nor $L\mu$ over Kripke structures
  - no game characterization, alternation-free fragment

The probabilistic μ-calculi zoo

- q m μ = quantitative interpretation of μ-calculus [HK97,MM97]
  - $\cup = \max$, $\cap = \min$, no PCTL, game characterization on finite models
- GPL = extension with finite nesting of $[\cdot]_{\succ p}$ quantifications [CPN99]
  - expresses PCTL* but neither $\exists\Box a$ nor $L\mu$ over Kripke structures
  - no game characterization, alternation-free fragment
- $pL\mu^{\odot}_{\oplus}$ is $L\mu+$ Lukasiewicz-operators + more [MS13]
  - probabilistic quantification = fixed point and multiplication
  - (tree) game characterization over all models, encodes PCTL

The probabilistic μ-calculi zoo

- qmμ = quantitative interpretation of μ-calculus [HK97,MM97]
  - $\cup$ = max, $\cap$ = min, no PCTL, game characterization on finite models
- GPL = extension with finite nesting of $[\cdot]_{\succ p}$ quantifications [CPN99]
  - expresses PCTL* but neither $\exists\Box a$ nor $L\mu$ over Kripke structures
  - no game characterization, alternation-free fragment
- $pL\mu^{\odot}_{\oplus}$ is $L\mu+$ Lukasiewicz-operators + more [MS13]
  - probabilistic quantification = fixed point and multiplication
  - (tree) game characterization over all models, encodes PCTL
- μ^p and μPCTL [CKP15]
  - distinguishes between qualitative and quantitative formulas
  - model checking μ^p-calculus is as hard as solving parity games
  - poly-time model checking of μPCTL for bounded alternation depth

The probabilistic μ-calculi zoo

- q$m\mu$ = quantitative interpretation of μ-calculus [HK97,MM97]
  - $\cup$ = max, $\cap$ = min, no PCTL, game characterization on finite models
- GPL = extension with finite nesting of $[\cdot]_{\succ p}$ quantifications [CPN99]
  - expresses PCTL* but neither $\exists\Box a$ nor $L\mu$ over Kripke structures
  - no game characterization, alternation-free fragment
- $pL\mu^{\odot}_{\oplus}$ is $L\mu+$ Lukasiewicz-operators + more [MS13]
  - probabilistic quantification = fixed point and multiplication
  - (tree) game characterization over all models, encodes PCTL
- μ^p and μPCTL [CKP15]
  - distinguishes between qualitative and quantitative formulas
  - model checking μ^p-calculus is as hard as solving parity games
  - poly-time model checking of μPCTL for bounded alternation depth
- PμTL = Lμ + $[\cdot]_{\succ p}$ for next-modalities [LSWZ15]
  - satisfiability by emptiness in prob. alt. parity automata (in 2EXPTIME)

The logic PμTL

Syntax

$$\Phi ::= a \mid \neg a \mid Z \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2 \mid [\bigcirc \Phi]_{\succ p} \mid \mu Z.\, \Phi \mid \nu Z.\, \Phi$$

$\mu Z.\, \Phi$ is valid for all states in the smallest set Z satisfying $Z = \Phi$.

The logic PμTL

Syntax

$$\Phi ::= a \mid \neg a \mid Z \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2 \mid [\bigcirc \Phi]_{\succ p} \mid \mu Z.\, \Phi \mid \nu Z.\, \Phi$$

$\mu Z.\, \Phi$ is valid for all states in the smallest set Z satisfying $Z = \Phi$.

Semantics

Let s be a Markov chain state with probability function $\mathbf{P}$:

$$\begin{array}{lll} s \models a & \text{iff} & a \in L(s) \\ \ldots & \ldots & \ldots\ldots \\ s \models [\bigcirc \Phi]_{\succ p} & \text{iff} & \sum\limits_{t:t\models\Phi} \mathbf{P}(s,t) \succ p \\ s \models \mu Z.\Phi & \text{iff} & s \in \bigcap\{S : \Phi(S) \subseteq S\} \\ s \models \nu Z.\Phi & \text{iff} & s \in \bigcup\{S : S \subseteq \Phi(S)\} \end{array}$$

where $\Phi(S)$ denotes formula Φ where Z is replaced by S.

Some facts about PμTL [CKP15,LSWZ15]

[4] On finite Markov chains.

Some facts about PμTL [CKP15,LSWZ15]

PμTL and PCTL are incomparable

1. $\nu Z. (a \wedge [\bigcirc Z]_{>0})$ is equivalent to CTL-formula $\exists\Box a$.
2. PCTL-formula $[\Diamond a]_{\geqslant 1/2}$ cannot be expressed in PμTL.

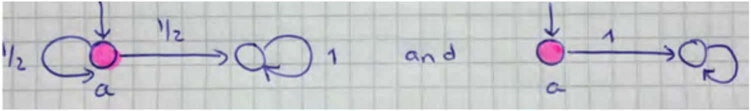
[4] On finite Markov chains.

Some facts about PμTL [CKP15,LSWZ15]

PμTL and PCTL are incomparable

1. $\nu Z. \left(a \wedge [\bigcirc Z]_{>0}\right)$ is equivalent to CTL-formula $\exists\Box a$.
2. PCTL-formula $[\Diamond a]_{\geqslant 1/2}$ cannot be expressed in PμTL.

Zero probabilities: PμTL-formula $\nu Z. \left(a \wedge [\bigcirc Z]_{\geqslant 1/2}\right)$ distinguishes



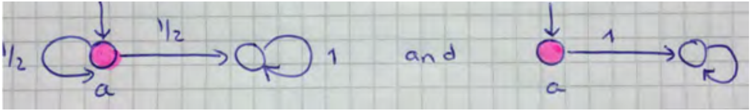
[4] On finite Markov chains.

Some facts about PμTL [CKP15,LSWZ15]

PμTL and PCTL are incomparable

1. $\nu Z.\,(a \wedge [\bigcirc\, Z]_{>0})$ is equivalent to CTL-formula $\exists\Box a$.
2. PCTL-formula $[\Diamond a]_{\geqslant 1/2}$ cannot be expressed in PμTL.

Zero probabilities: PμTL-formula $\nu Z.\left(a \wedge [\bigcirc\, Z]_{\geqslant 1/2}\right)$ distinguishes

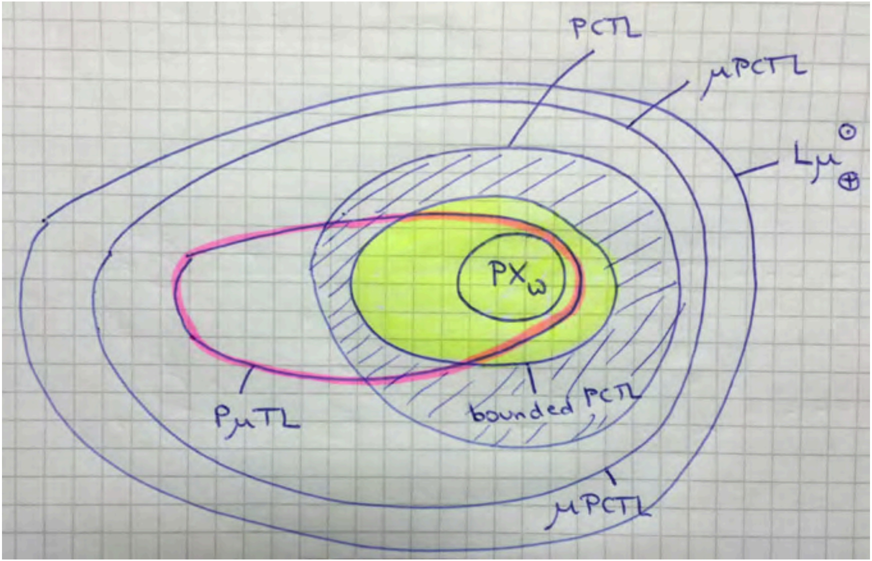


Qualitative PμTL strictly contains qualitative PCTL

qPCTL-formulas can be transformed into equivalent qPμTL ones[4].

[4] On finite Markov chains.

Expressiveness



What is new about PμTL satisfiability?

Our results:

1. A satisfiable PμTL-formula has a bounded-degree model in $\mathcal{O}(|\Phi|)$.
2. PμTL has the finite model property with size exponential in $|\Phi|$.
3. A satisfiable PμTL-formula has a rational model.
4. Satisfiability for PμTL is as hard as satisfiability for Lμ.

What is new about PμTL satisfiability?

Our results:

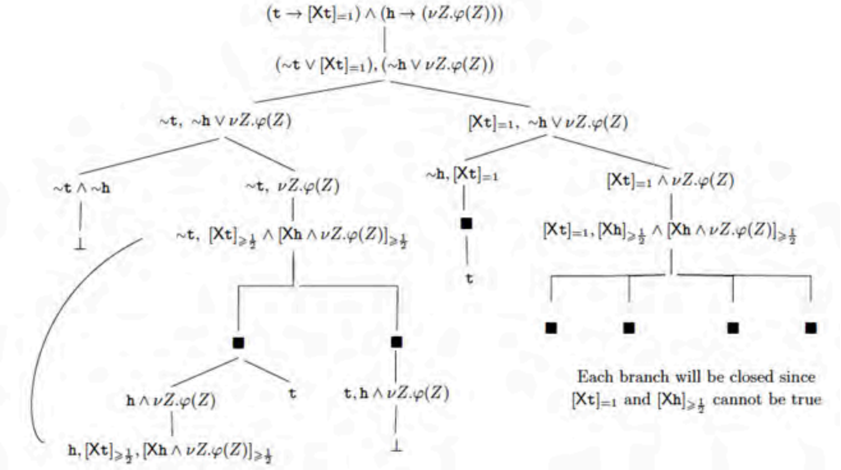
1. A satisfiable PμTL-formula has a bounded-degree model in $\mathcal{O}(|\Phi|)$.
   - adapted tableaux-like techniques à la Lμ to probabilistic setting
2. PμTL has the finite model property with size exponential in $|\Phi|$.
3. A satisfiable PμTL-formula has a rational model.
4. Satisfiability for PμTL is as hard as satisfiability for Lμ.

What is new about PμTL satisfiability?

Our results:

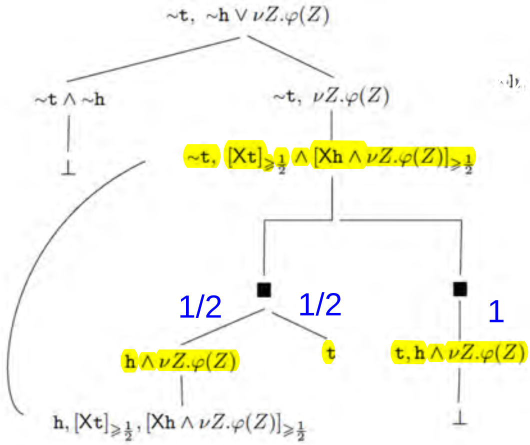
1. A satisfiable PμTL-formula has a bounded-degree model in $\mathcal{O}(|\Phi|)$.
2. PμTL has the finite model property with size exponential in $|\Phi|$.
   - using stochastic parity game = game graph $\times$ DPA
   - key is the distribution of formulas as weighted covers in the game
3. A satisfiable PμTL-formula has a rational model.
   - the weight functions used in the game are all rational
4. Satisfiability for PμTL is as hard as satisfiability for Lμ.

Example satisfiability game



where $\varphi(Z) = [\bigcirc \mathsf{t}]_{\geqslant 1/2} \wedge [\bigcirc \mathsf{h} \wedge Z]_{\geqslant 1/2}$

Weighted cover in satisfiability game



where $\varphi(Z) = [\bigcirc \mathsf{t}]_{\geqslant 1/2} \wedge [\bigcirc \mathsf{h} \wedge Z]_{\geqslant 1/2}$

Epilogue

Take-home messages:

1. Complexity and FMP for two quantitative sub-logics of PCTL
   - probabilistically quantified (unbounded) nested next-modalities
   - probabilistically quantified bounded until-operator
2. Complexity, FMP and RMP for a simple probabilistic μ-calculus
   2.1 probabilistically quantified next-modalities

Future work:

- Sound and complete axiomatizations [DFHM16]
- Satisfiability for other PCTL fragments and probabilistic μ-calculi
- Qualitative extensions of MSO [B16]

Full details can be found in arXiv paper version.