Analyzing Expected Outcomes and (Positive) Almost-Sure Termination of Probabilistic Programs is Hard

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How hard is it to solve these analysis problems?

Dissent in the Literature

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[Esparza *et al.* 2012]

"[Ordinary] termination is a purely topological property [...], but almost–sure termination is not.
[...] proving almost–sure termination requires arithmetic reasoning not offered by termination provers."

The Arithmetical Hierarchy

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- Class Σ_n^0 is defined as

$$\Sigma_n^0 = \Big\{ A \;\Big|\; A = \big\{ \vec{x} \;\big|\; \exists y_1 \, \forall y_2 \, \exists y_3 \, \cdots \, \exists / \forall y_n : (\vec{x}, y_1, y_2, y_3, \ldots, y_n) \in R \big\}, R \text{ is a decidable relation} \Big\}$$

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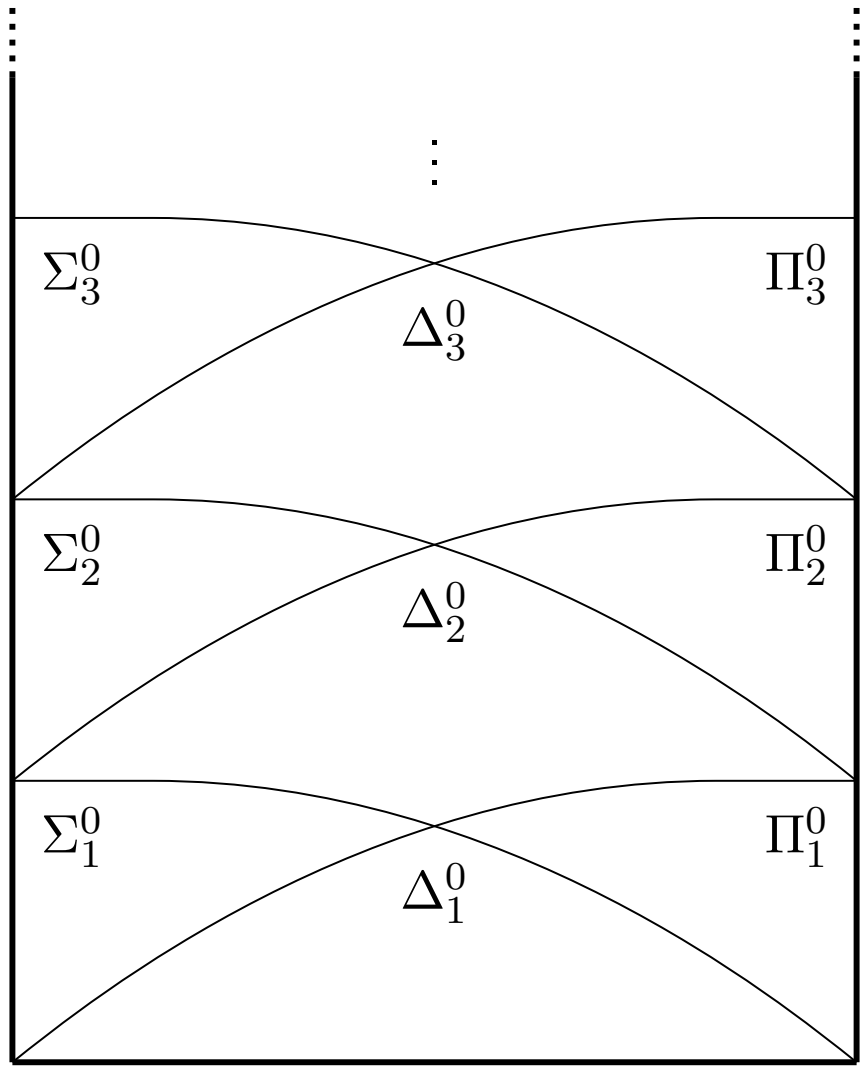
- Class Δ_n^0 is defined as $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$

The Arithmetical Hierarchy — The Bigger Picture

The following inclusion diagram holds (all inclusions are strict):



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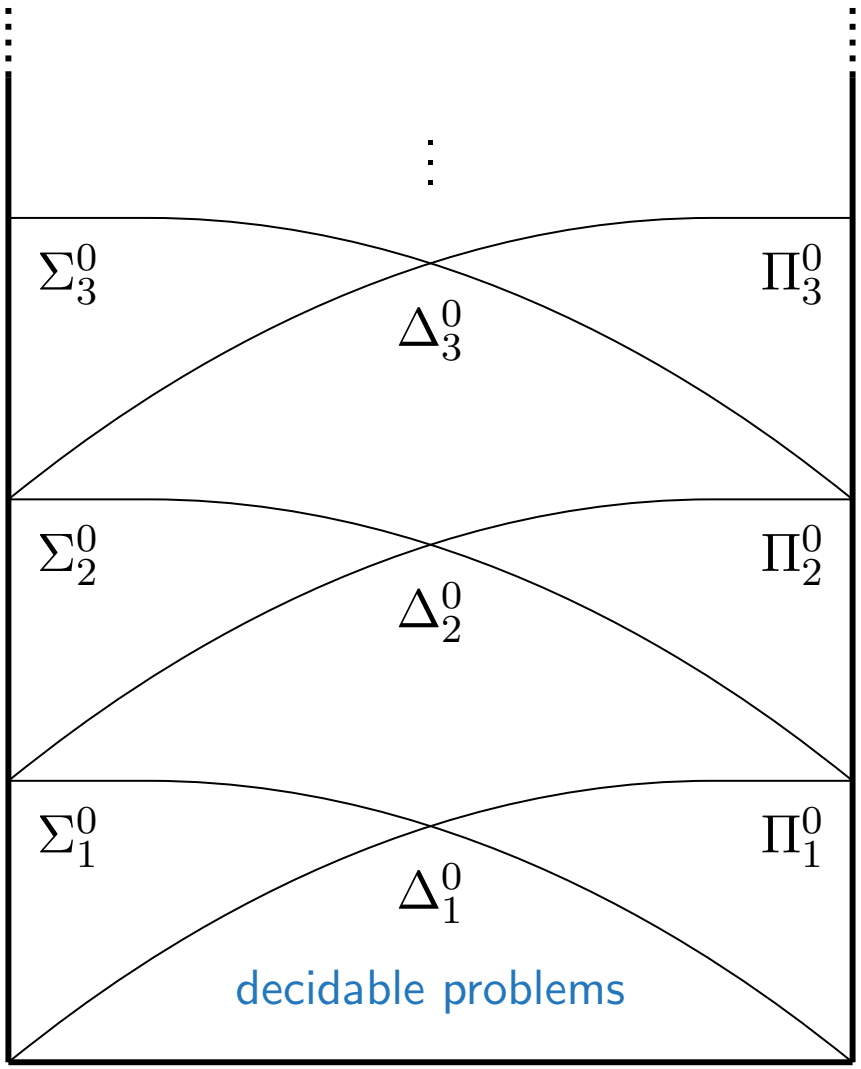

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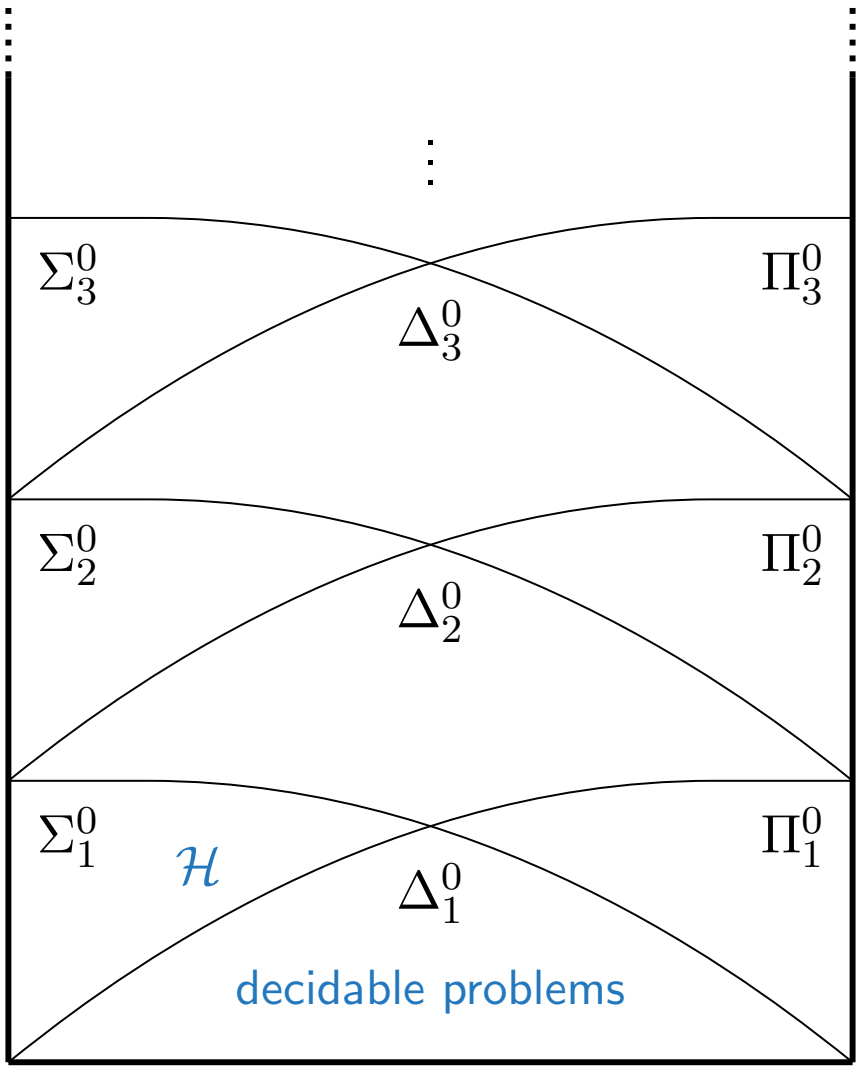

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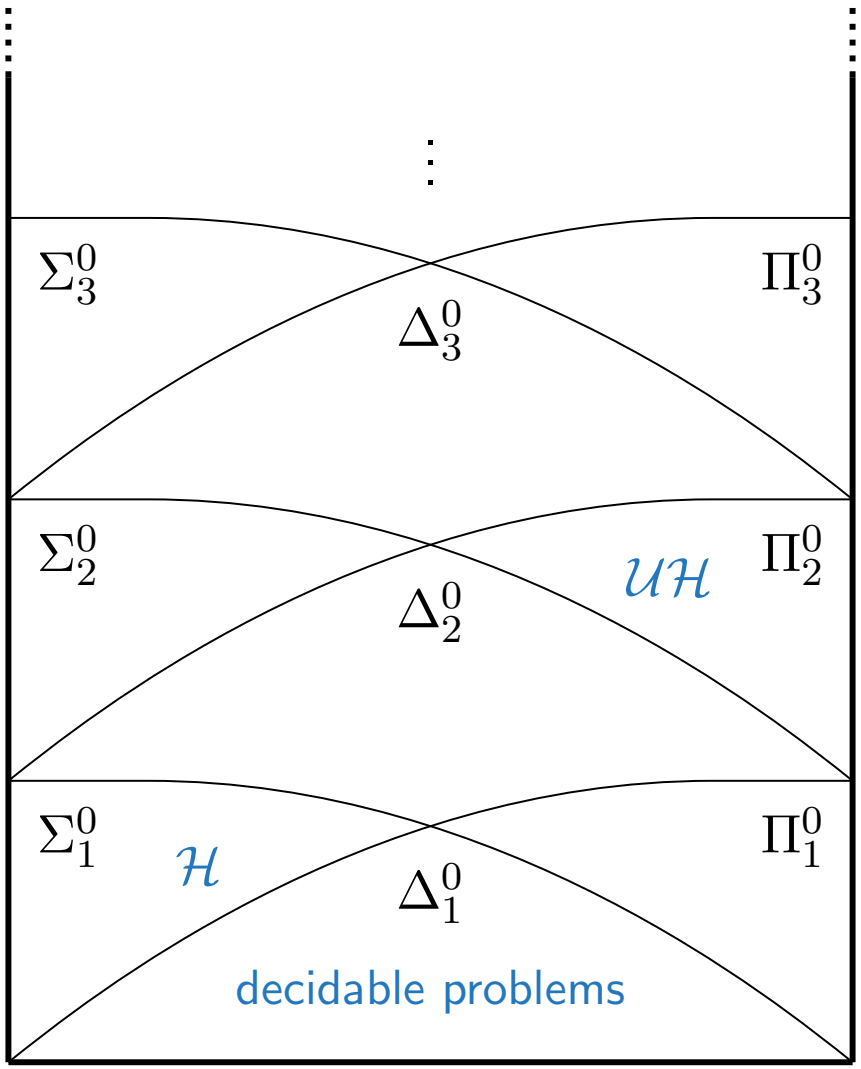

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- The expected number of steps until P terminates on input η:

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Lower and Upper Bounds, and Exact Expected Outcomes

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\begin{aligned}
(P, v, q) \in \mathcal{LEXP} &\iff q < \mathsf{E}_P(v) \\
(P, v, q) \in \mathcal{UEXP} &\iff q > \mathsf{E}_P(v) \\
(P, v, q) \in \mathcal{EXP} &\iff q = \mathsf{E}_P(v)
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Almost–Sure Termination $\mathcal{AST}$

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(P, \eta) \in \mathcal{AST} :\Longleftrightarrow \mathsf{Pr}_{P,\eta}(\downarrow) = 1
$$

Variations of $\mathcal{AST}$

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Positive Almost–Sure Termination $\mathcal{PAST}$

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Universal Versions of $\mathcal{AST}$ and $\mathcal{PAST}$

$$P \in \mathcal{UAST} \;:\Longleftrightarrow\; \forall \eta\colon (P, \eta) \in \mathcal{AST}$$
$$P \in \mathcal{UPAST} \;:\Longleftrightarrow\; \forall \eta\colon (P, \eta) \in \mathcal{PAST}$$

A (very) Simple Example Program

Consider the program P_{geo}:

```
x := 0;
{continue := 0} [0.5] {continue := 1};
while (continue ≠ 0){
    x := x + 1;
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- $\mathsf{E}_{P_{geo}}(\texttt{continue}) = 0$
- P_{geo} terminates almost–surely on all inputs
- Expected runtime of P_{geo} is $\mathcal{O}\left(\mathsf{E}_{P_{geo}}(x)\right)$ on all inputs

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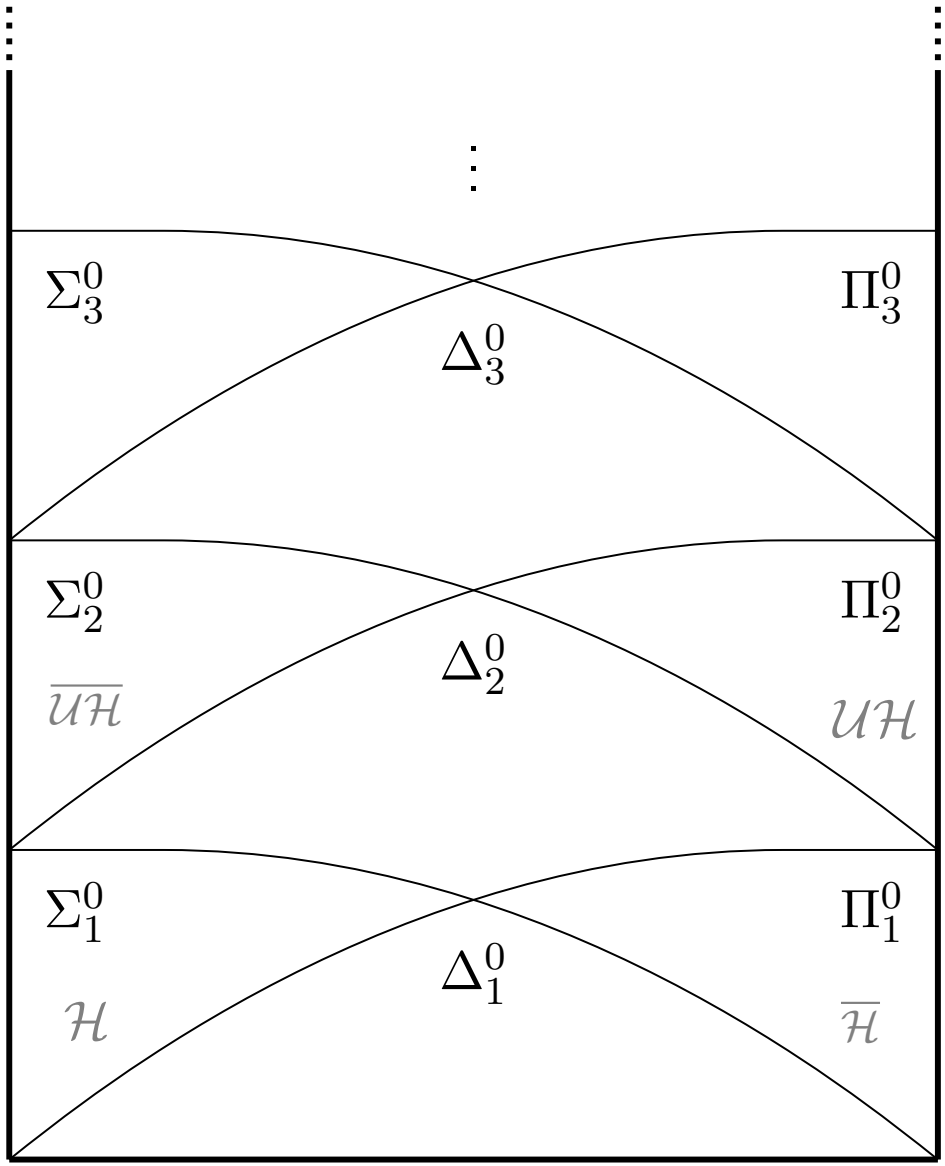

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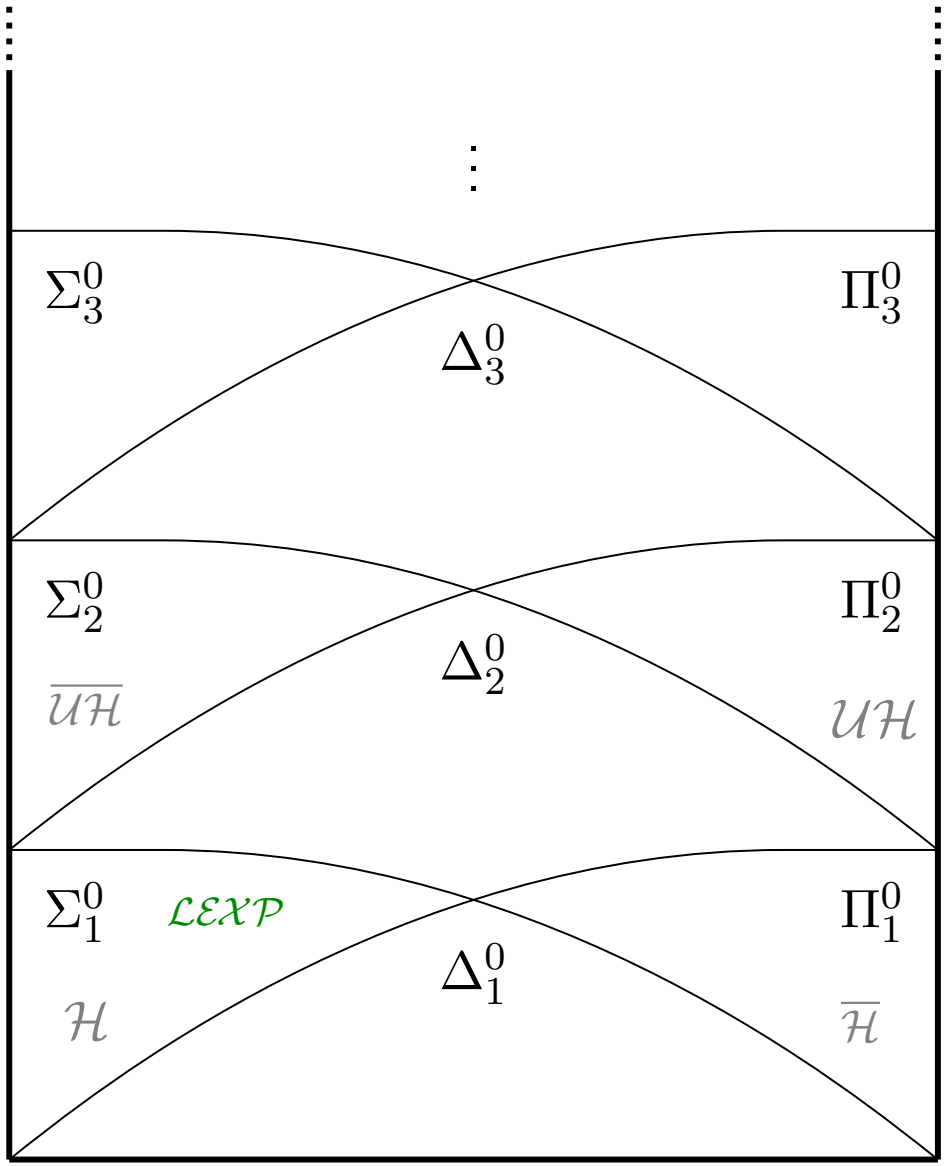

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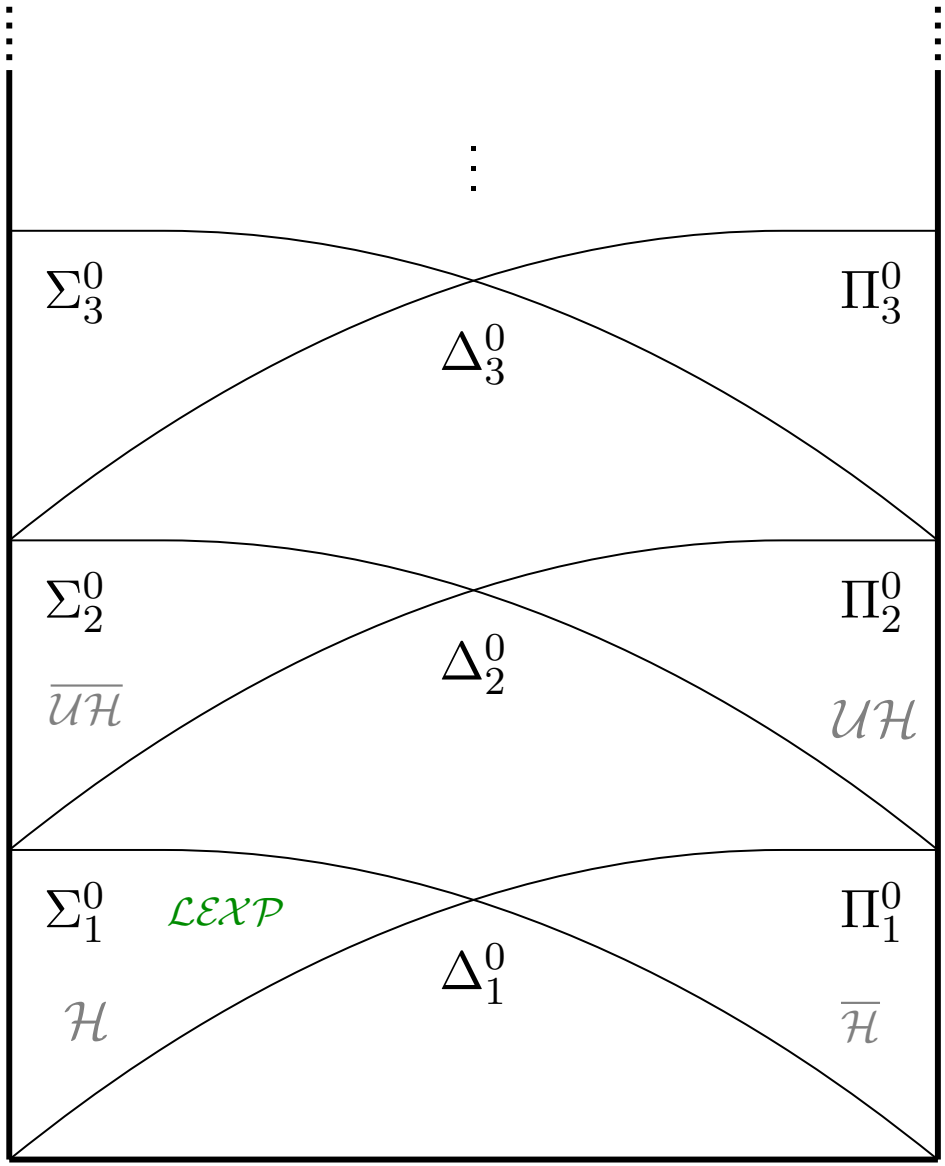

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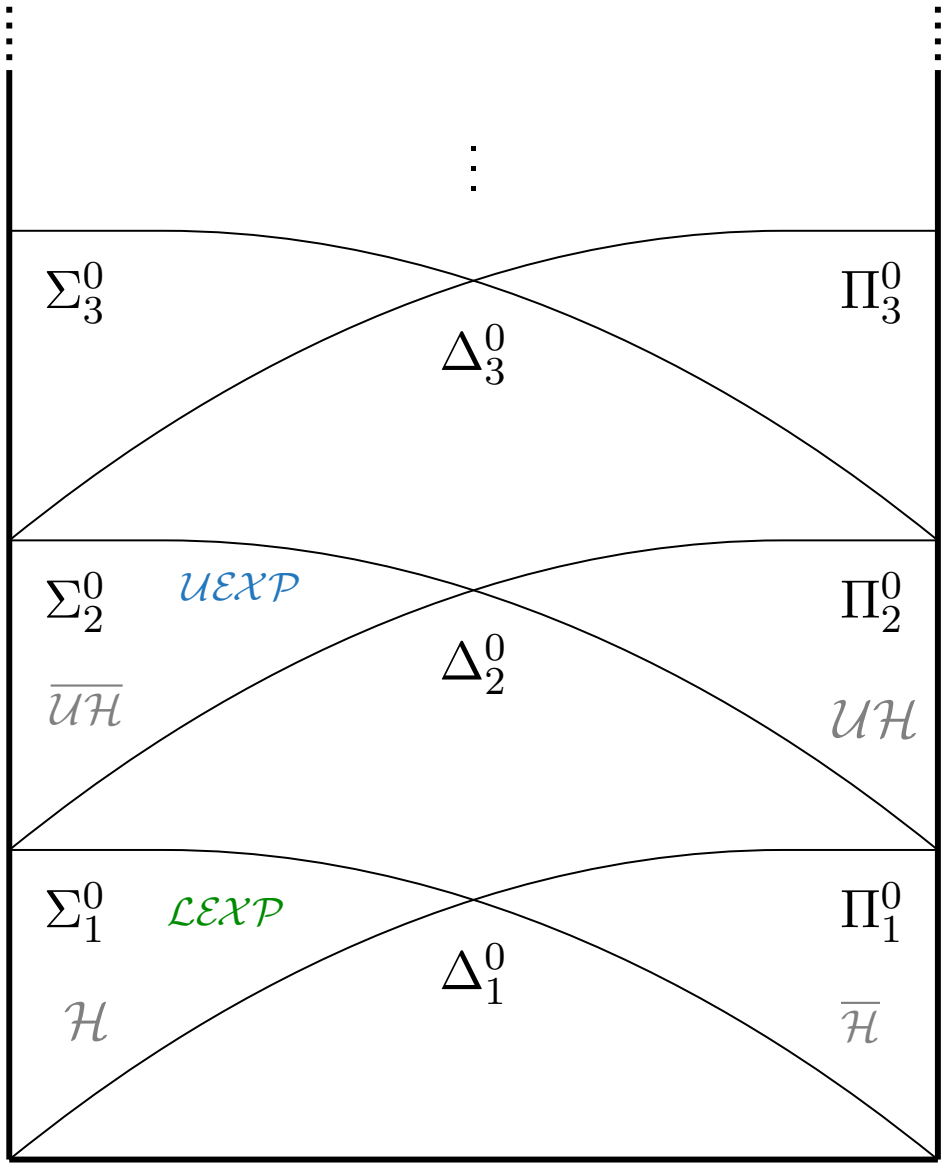

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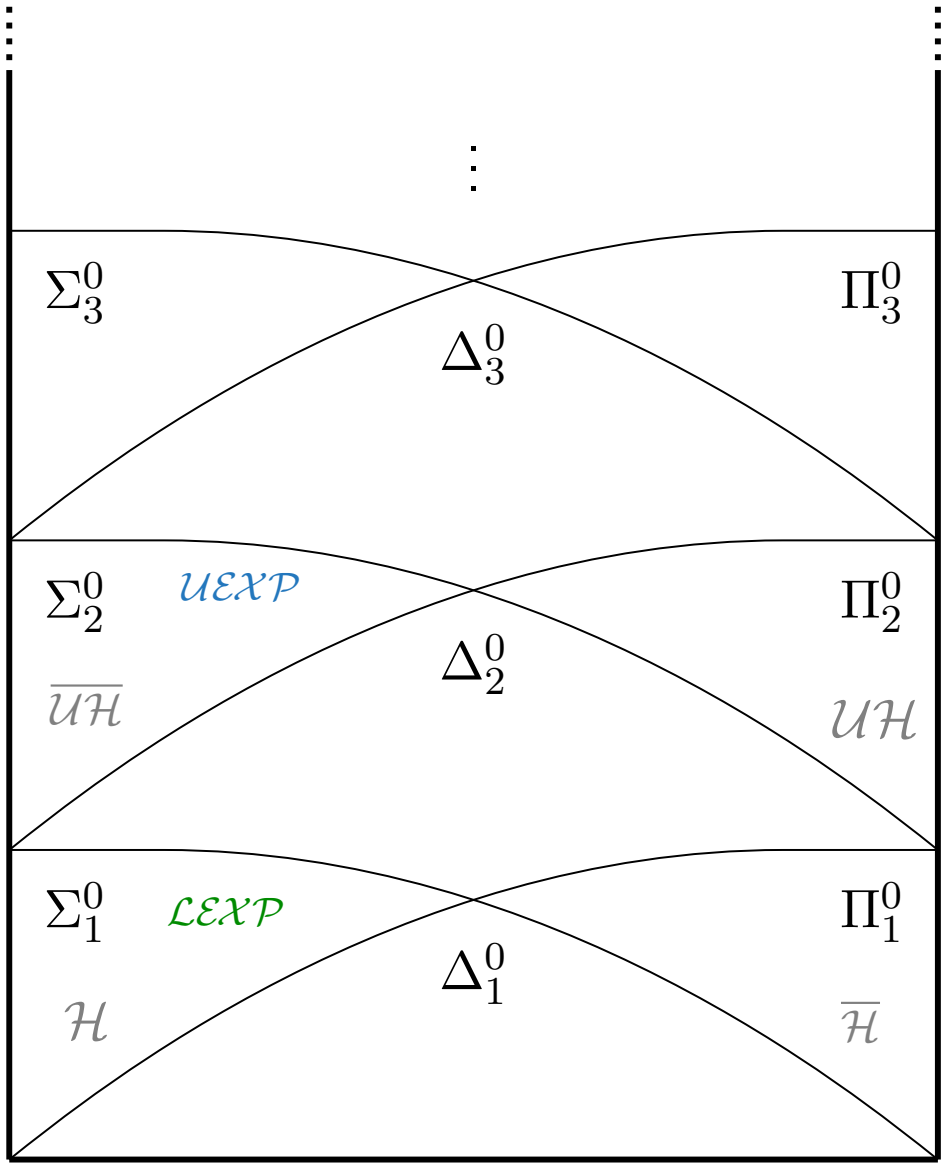

with access to $\mathcal{H}$–oracle: semi–decidable

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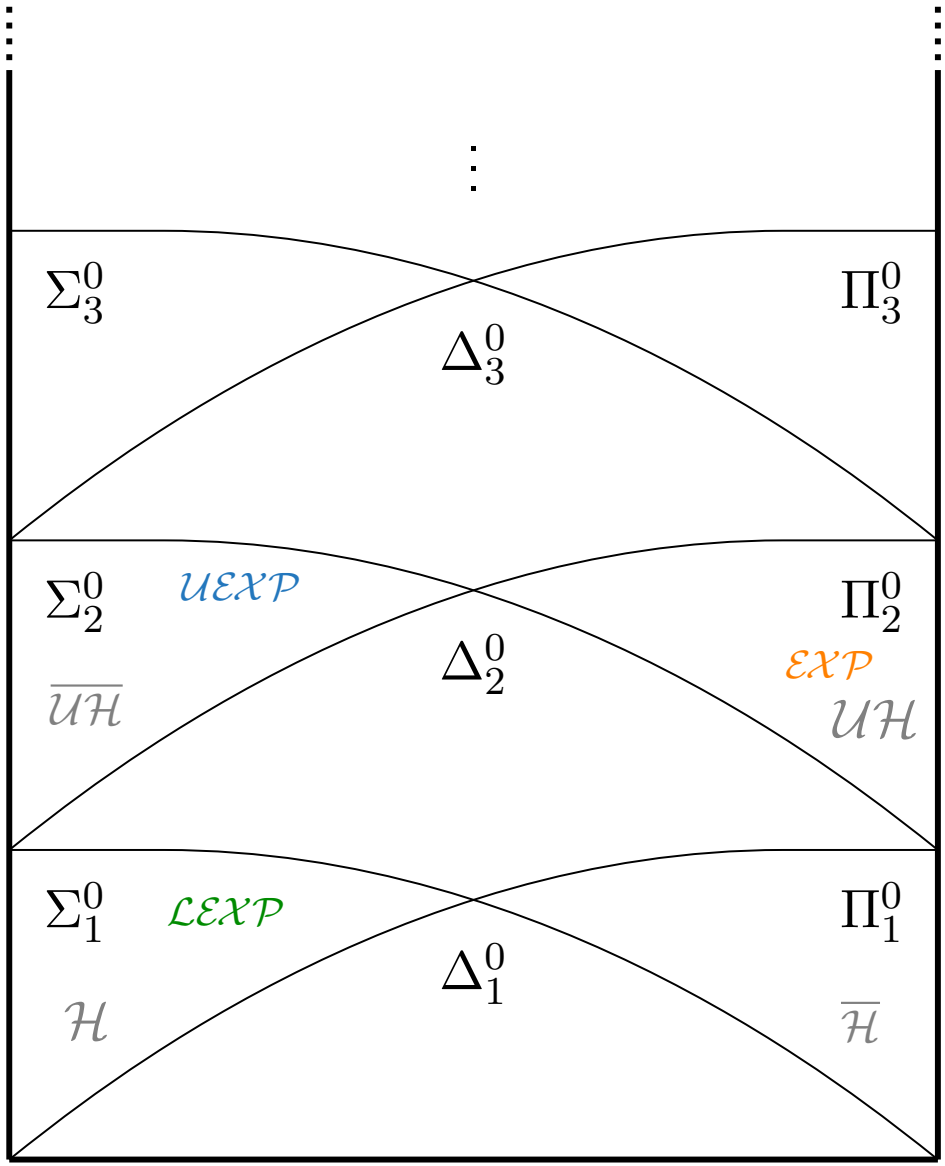

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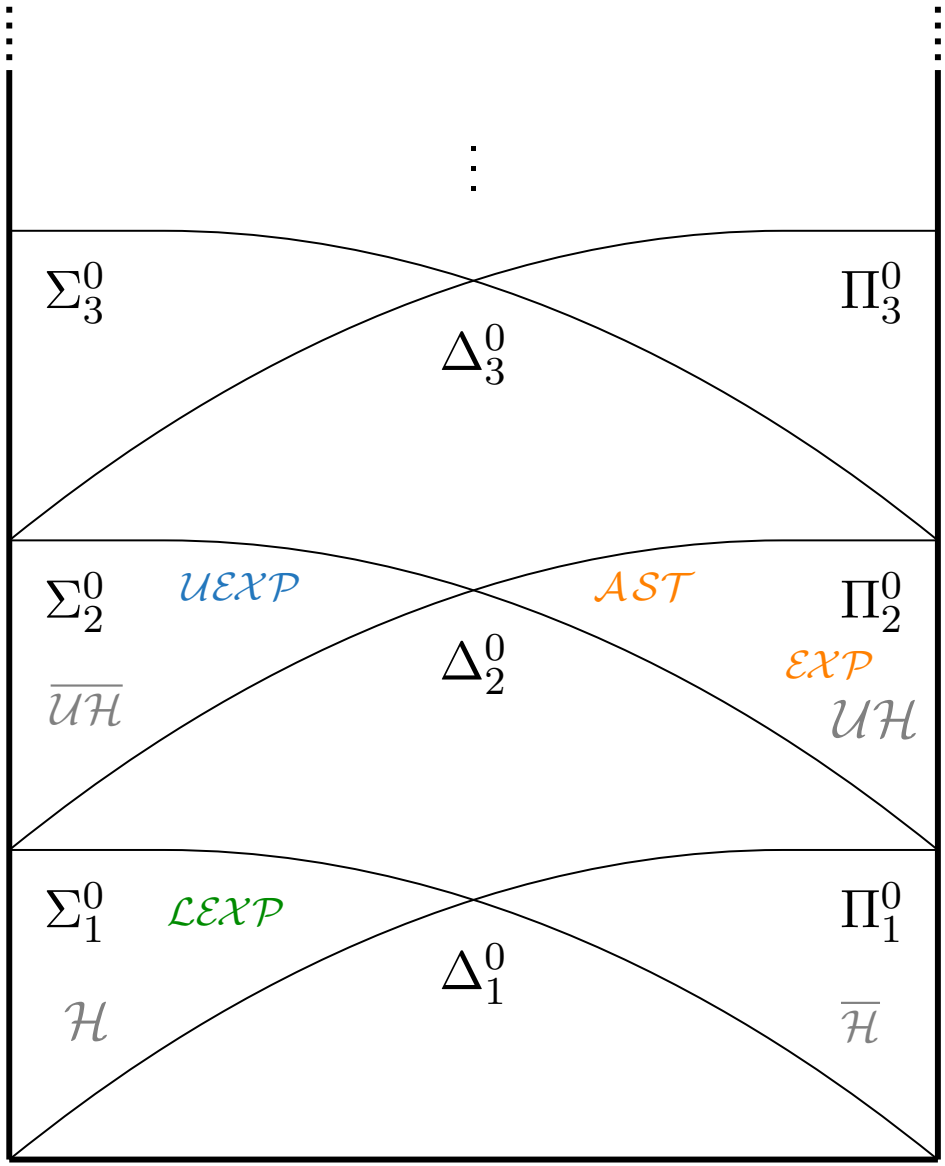

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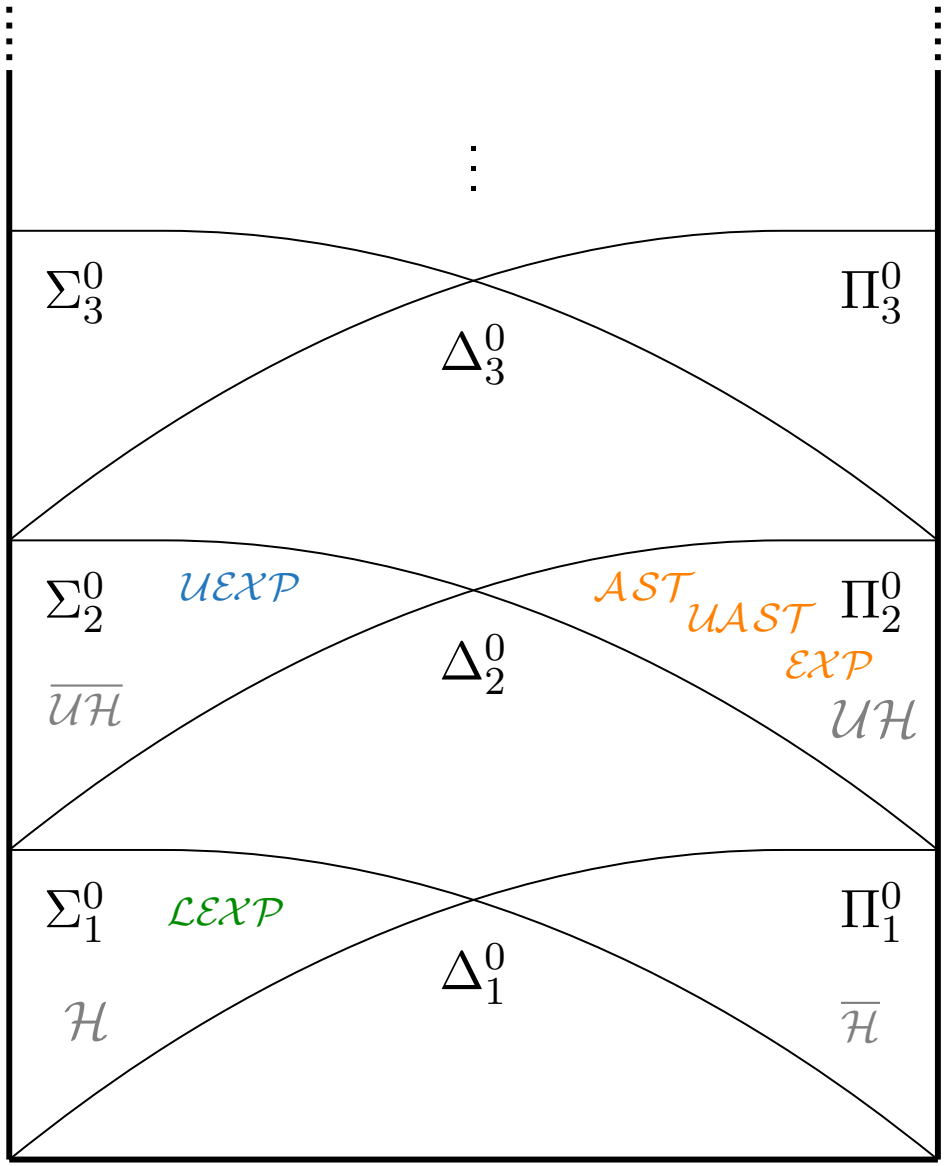

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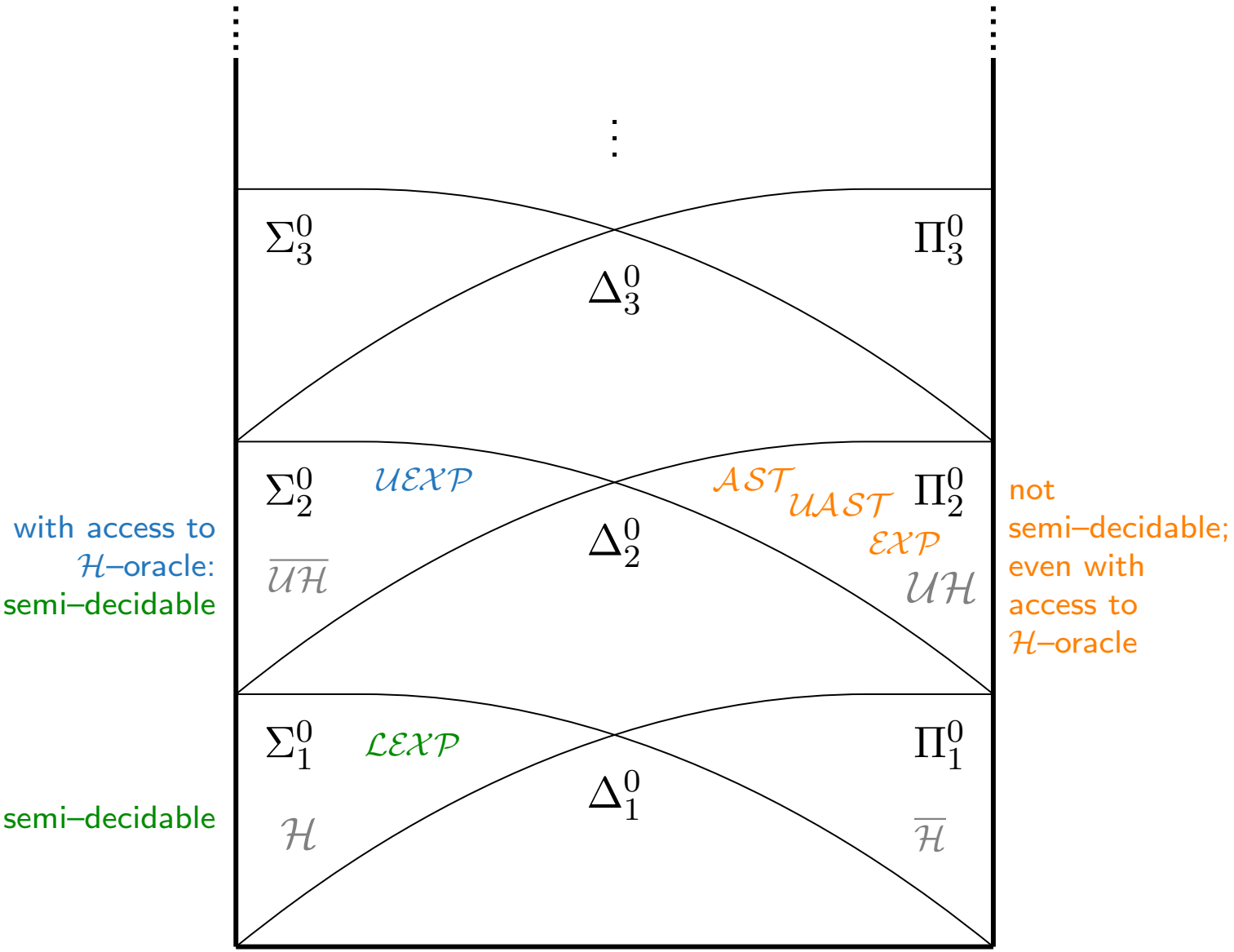

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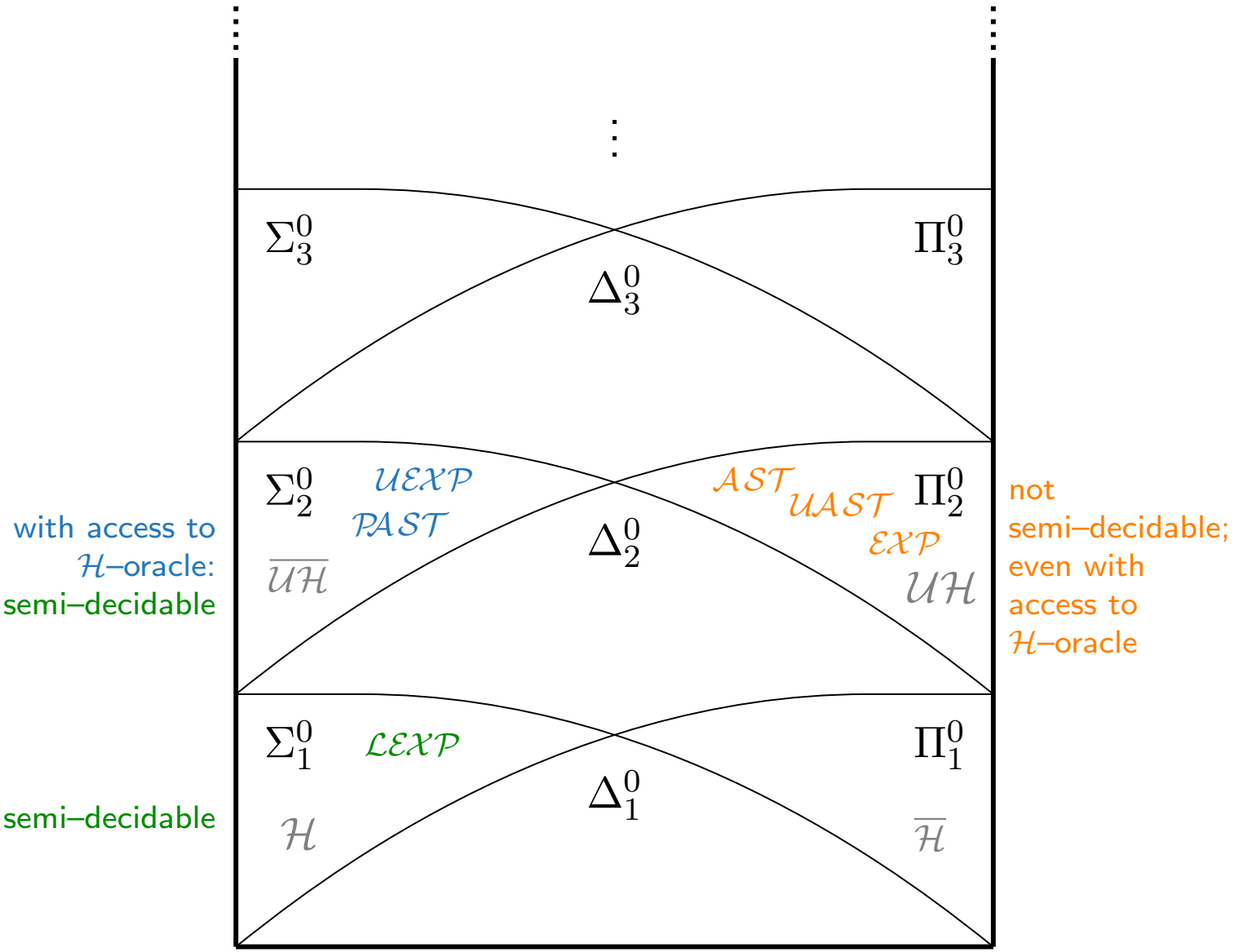

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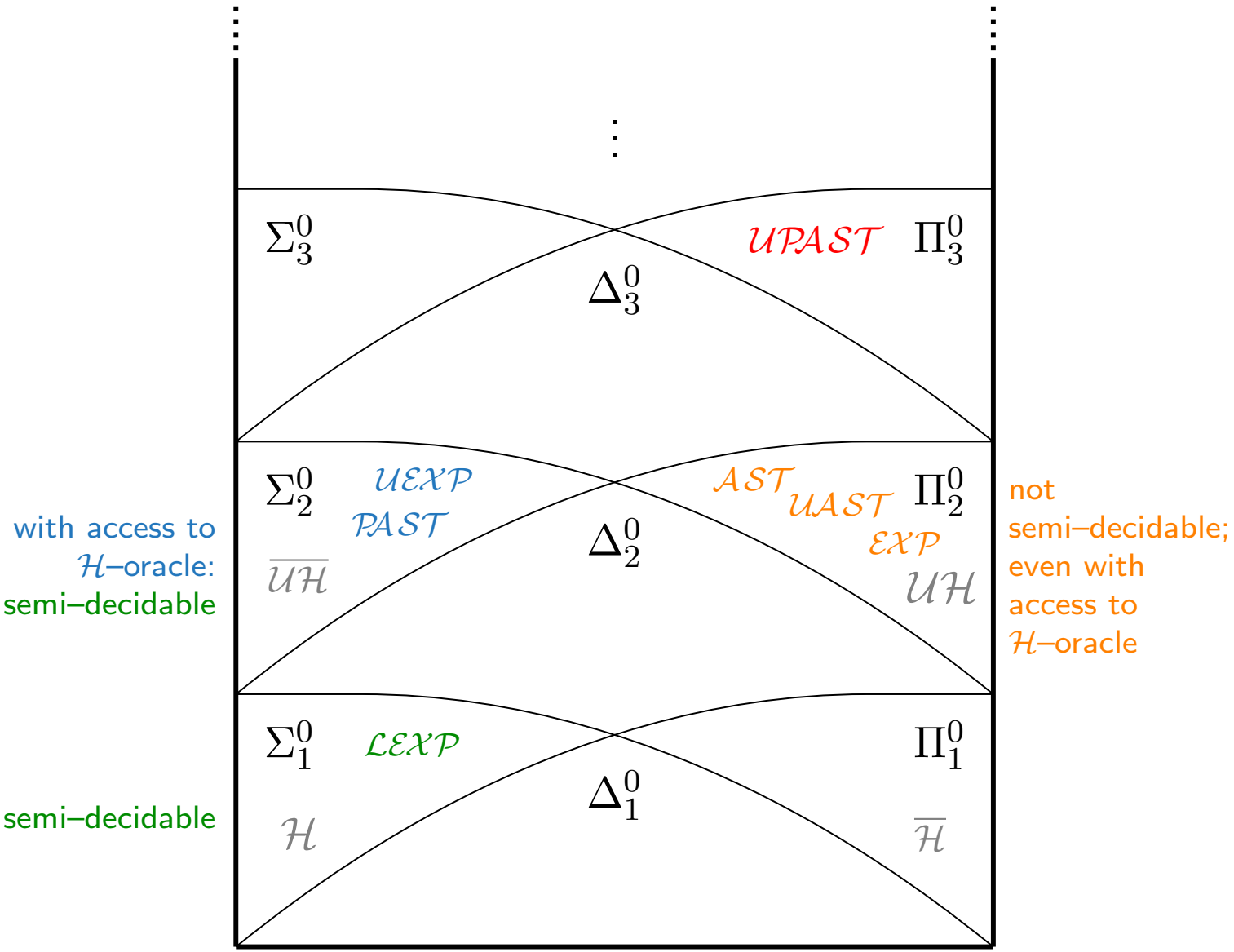

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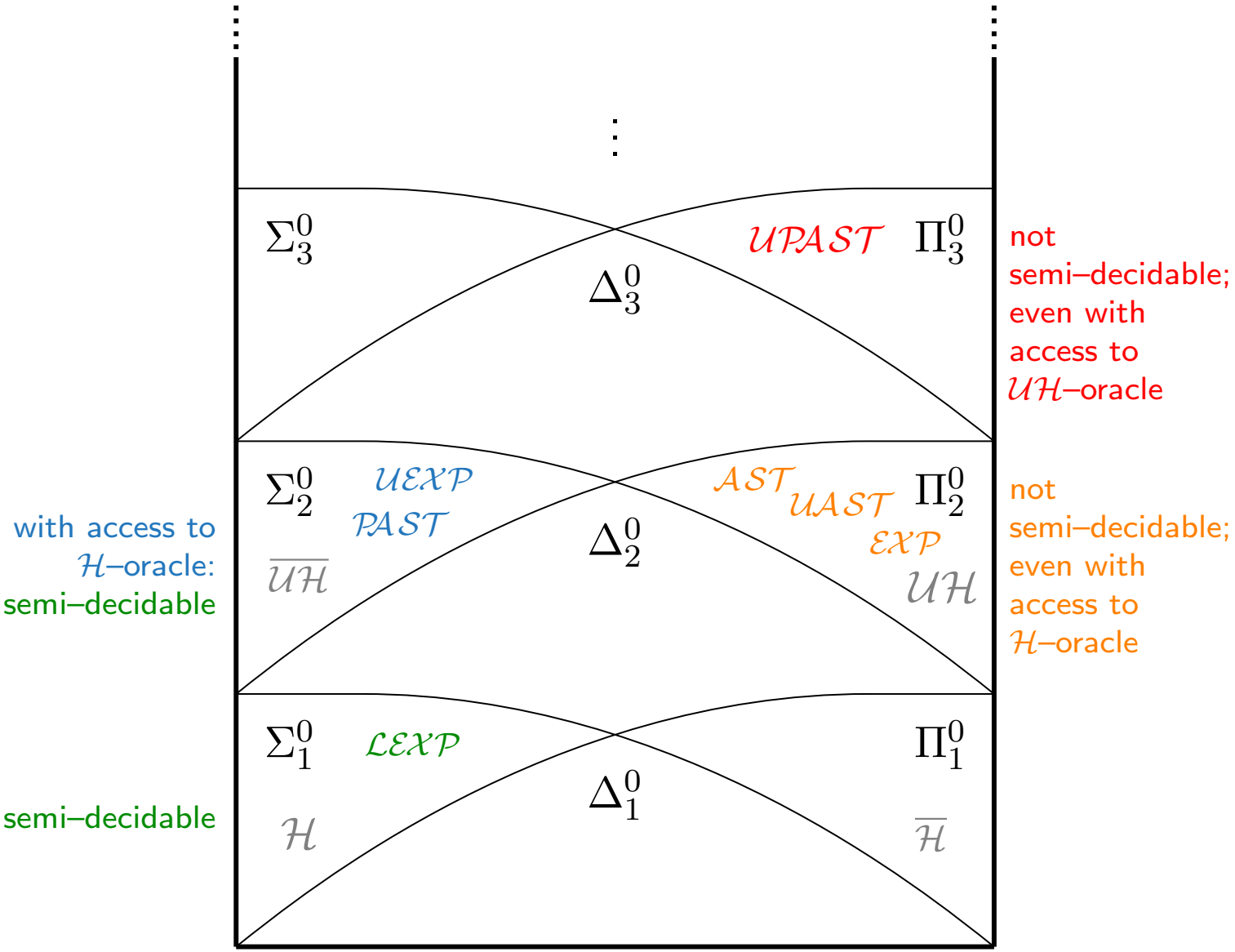

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