On the Hardness of Almost–Sure Termination

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Our Setting: Probabilistic Programs

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How hard is it to solve these analysis problems?

Dissent in the Literature

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Decision Problems We Studied

Lower and Upper Bounds, and Exact Expected Outcomes

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$$(P, \eta, v, q) \in \mathcal{EXP} \quad :\Longleftrightarrow \quad \text{“The e. v. of } v \text{ after termination of } P \text{ on input } \eta \text{ is } = q\text{”}$$

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Universal Versions of $\mathcal{AST}$ and $\mathcal{PAST}$

$$P \in \mathcal{UAST} \quad :\Longleftrightarrow \forall \eta\colon (P, \eta) \in \mathcal{AST}$$
$$P \in \mathcal{UPAST} \quad :\Longleftrightarrow \forall \eta\colon (P, \eta) \in \mathcal{PAST}$$

Hardness Results

The Arithmetical Hierarchy

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- Class Σ_n^0 is defined as

$$\Sigma_n^0 = \Big\{ A \;\Big|\; A = \{ \vec{x} \mid \exists y_1 \, \forall y_2 \, \exists y_3 \, \cdots \exists/\forall y_n : (\vec{x}, y_1, y_2, y_3, \ldots, y_n) \in R \}, R \text{ is a decidable relation} \Big\}$$

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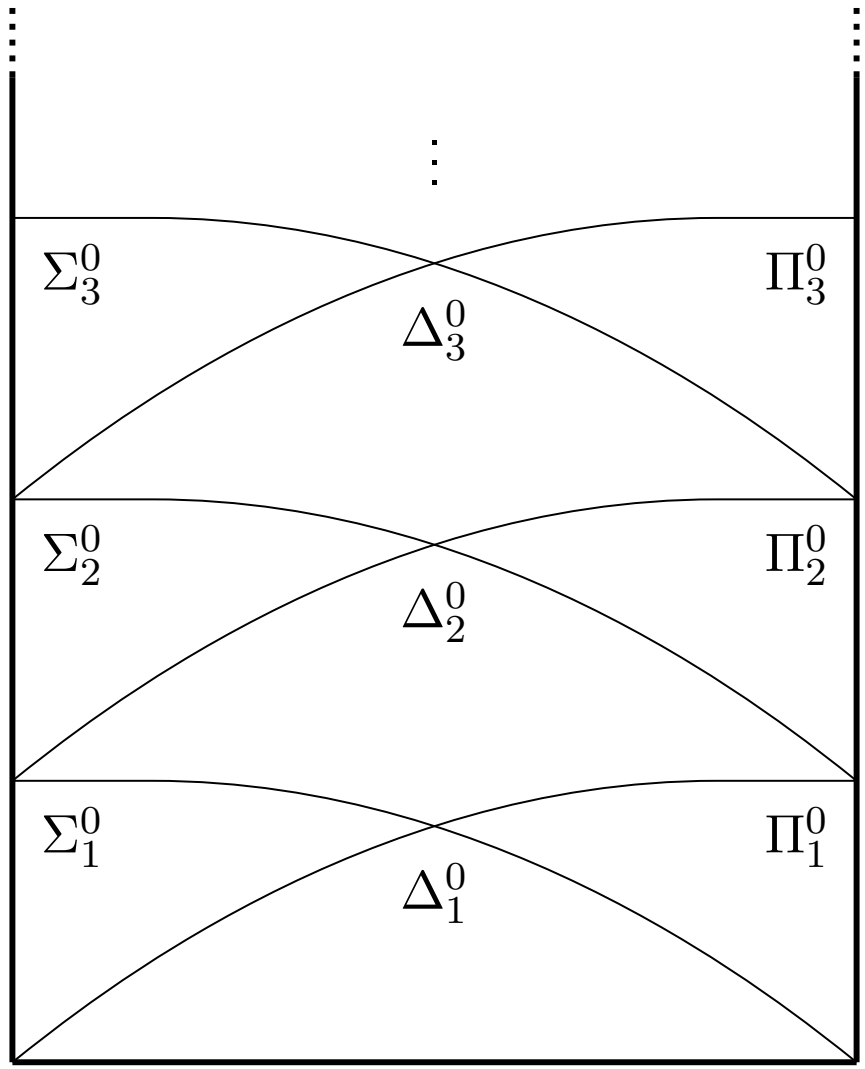
- Class Δ_n^0 is defined as $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$

The Arithmetical Hierarchy — The Bigger Picture

The following inclusion diagram holds (all inclusions are strict):



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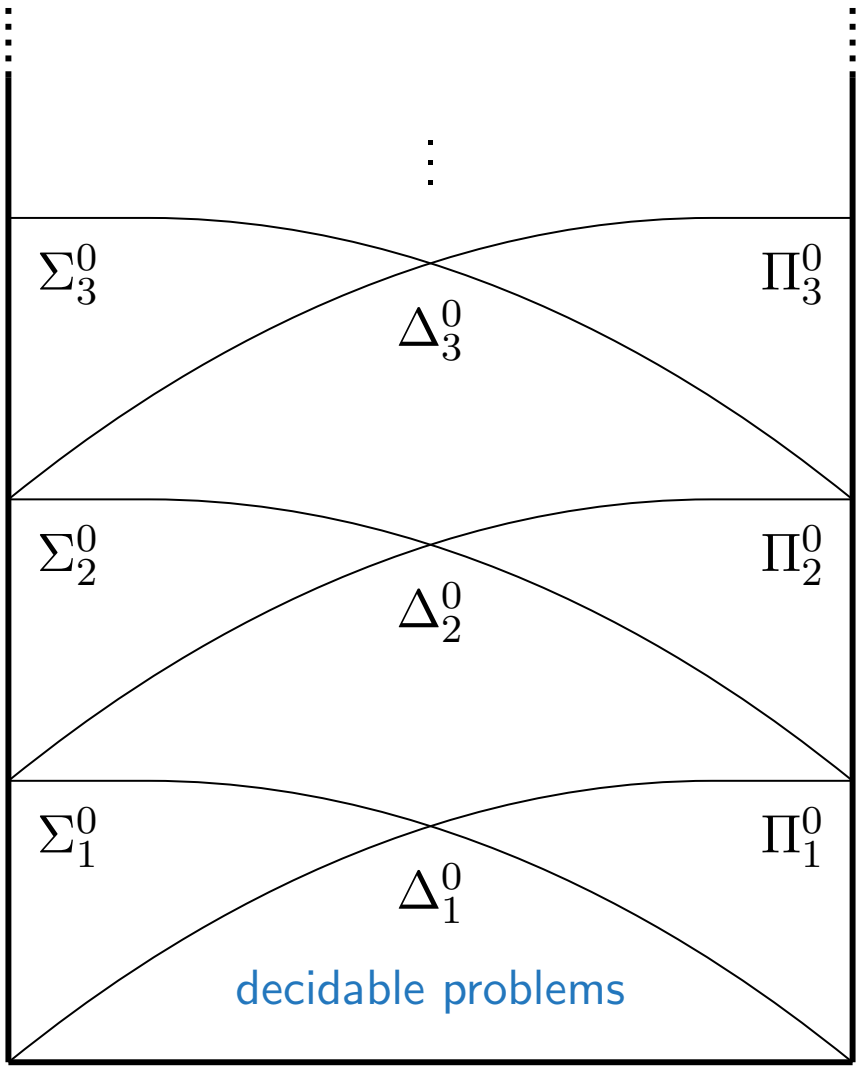

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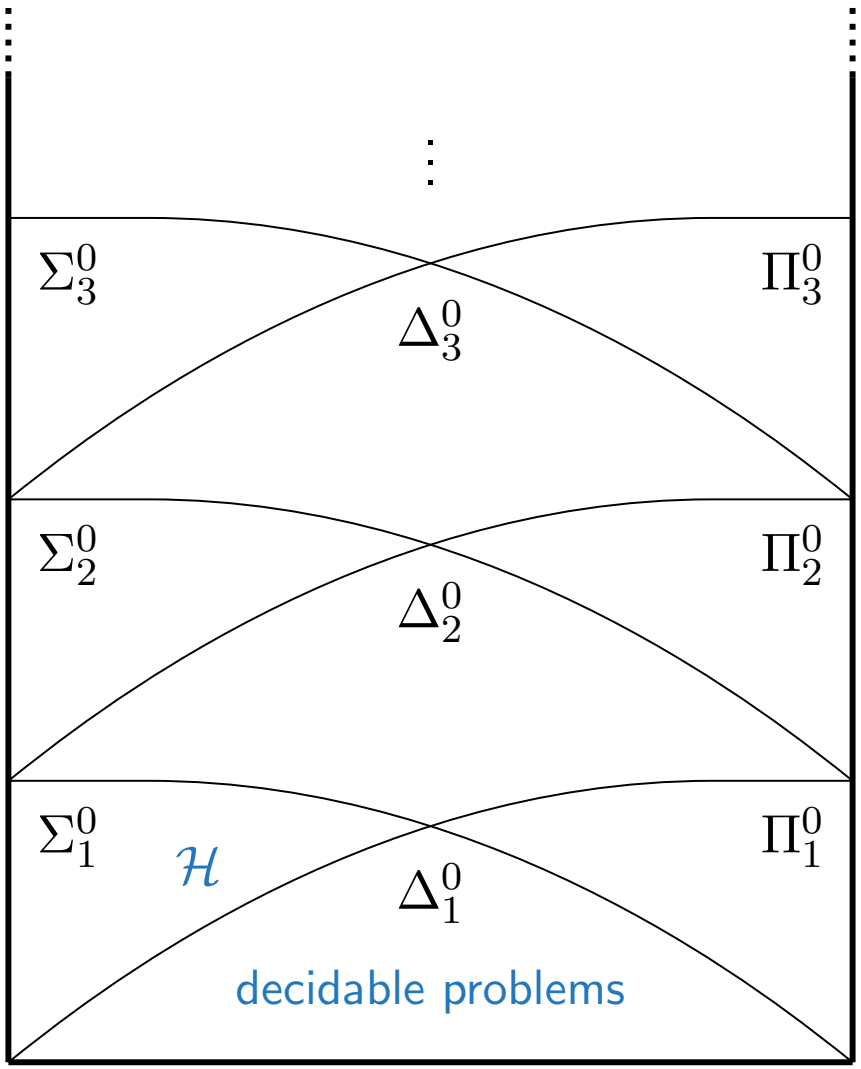

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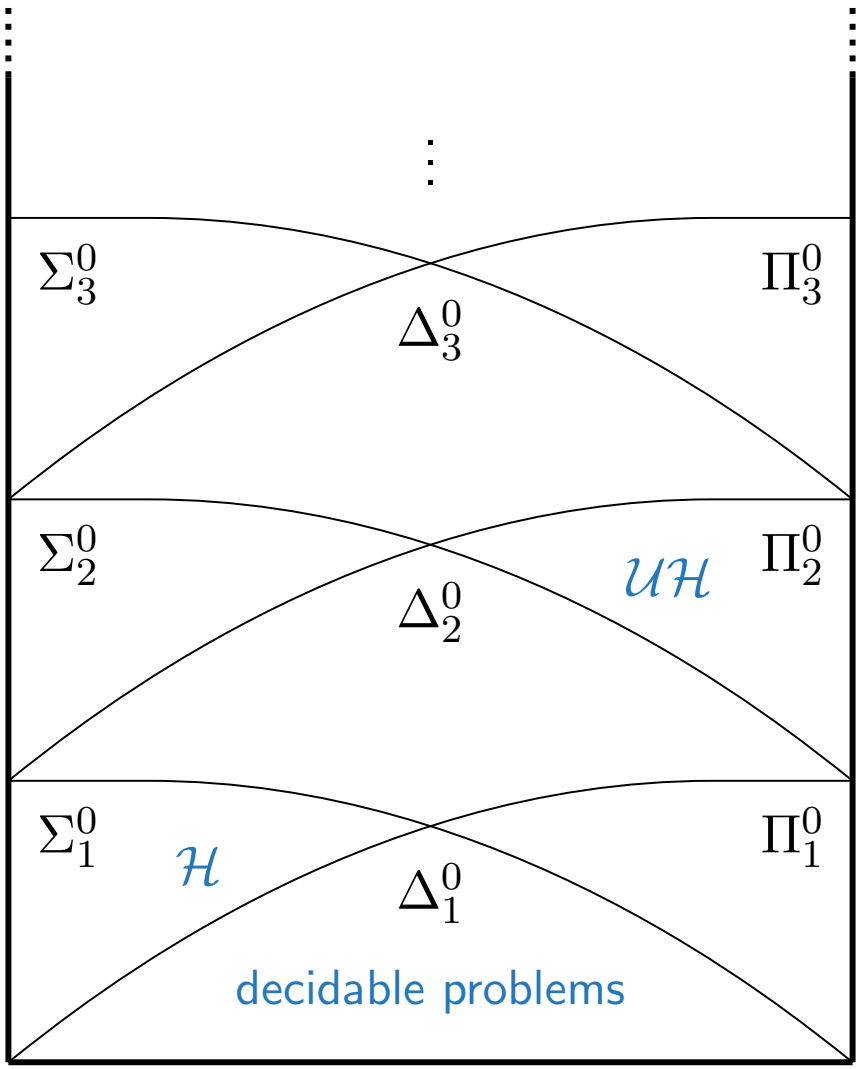

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Hardness of Deciding Positive Almost–Sure Termination

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Proof: Prove $\mathcal{PAST} \in \Sigma_2^0$ and prove $\overline{\mathcal{UH}} \leq_{\mathsf{m}} \mathcal{PAST}$.

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Reduction function computes from every ordinary program Q that does *not* terminate on all inputs, a probabilistic program P (depending on Q) and an input η,

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Reduction function computes from every ordinary program Q that does *not* terminate on all inputs, a probabilistic program P (depending on Q) and an input η, such that P does terminate in an expected finite number of steps on input η.

The Construction for $\overline{\mathcal{UH}} \leq \mathcal{PAST}$

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continue := 1;
while (continue ≠ 0){
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where $g_Q \colon \mathbb{N} \to \mathit{Inputs}_Q$ is an enumeration of all inputs for Q

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        Take 2^flips (meaningless) steps;
      } else {
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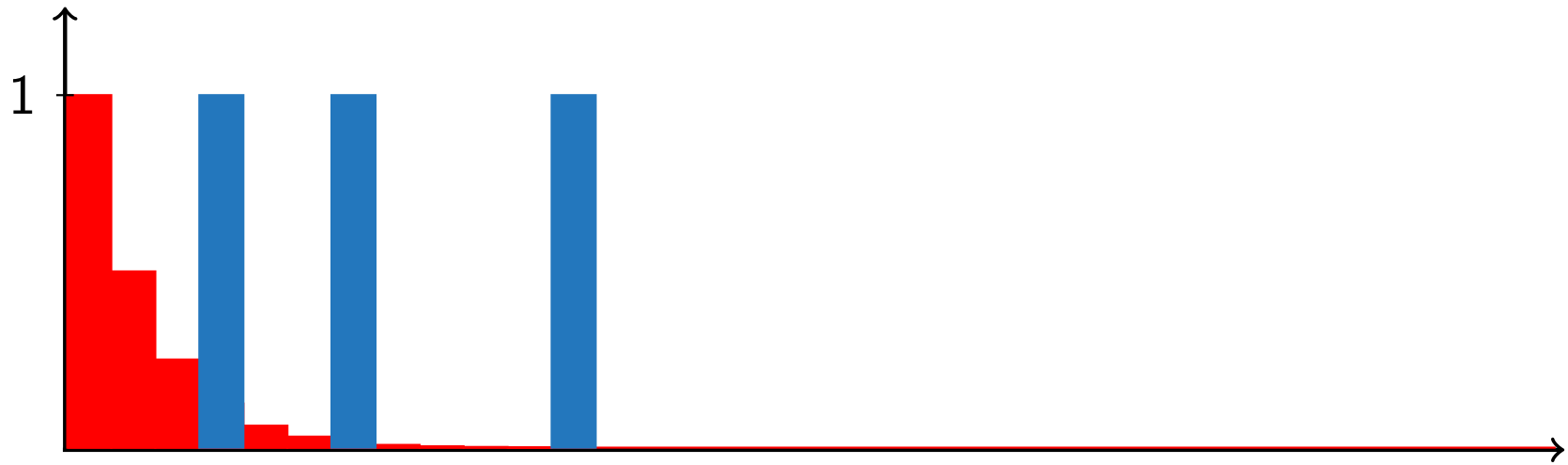

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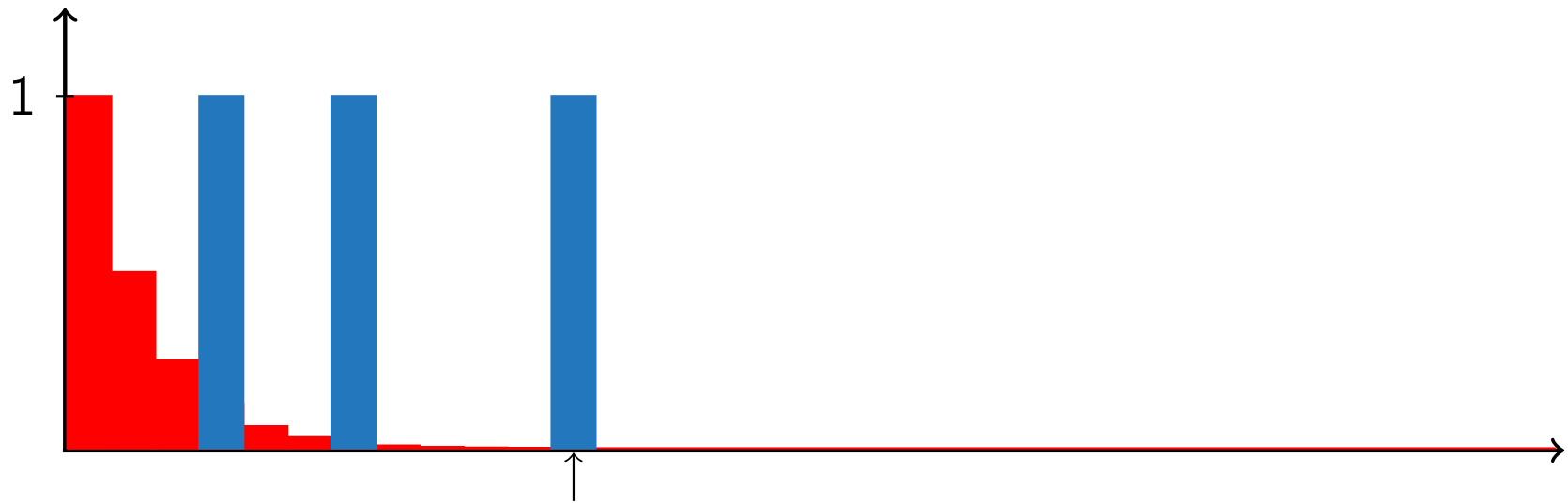

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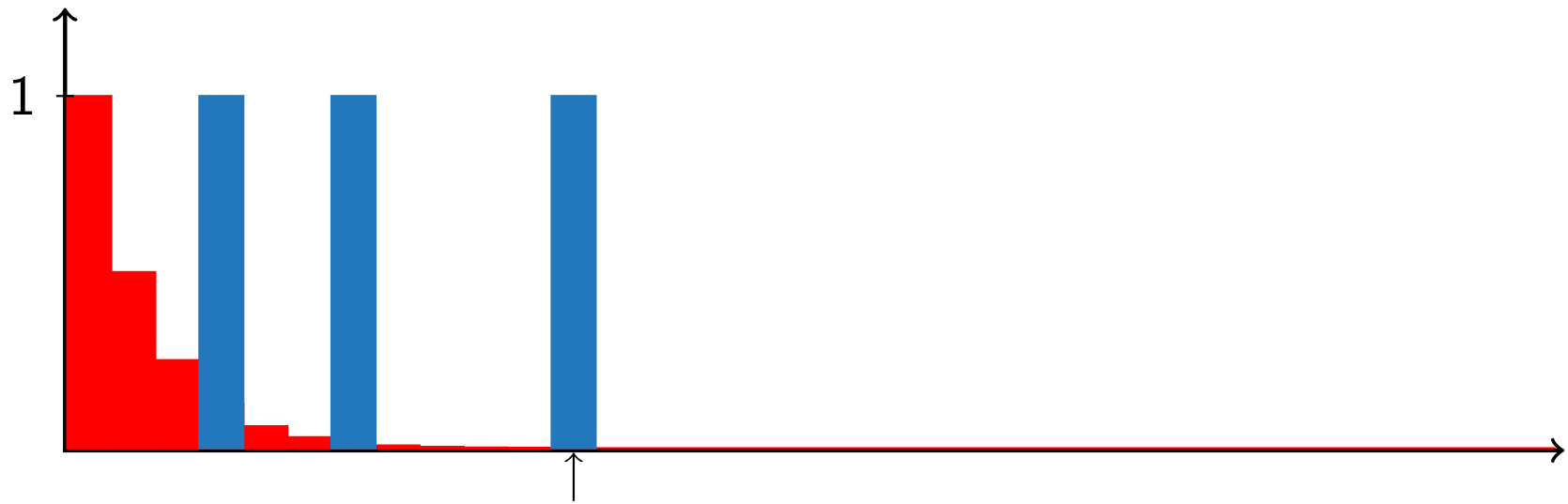

cheering on termination on $(i-1)$-th input

Finite cheering — finite expected runtime!

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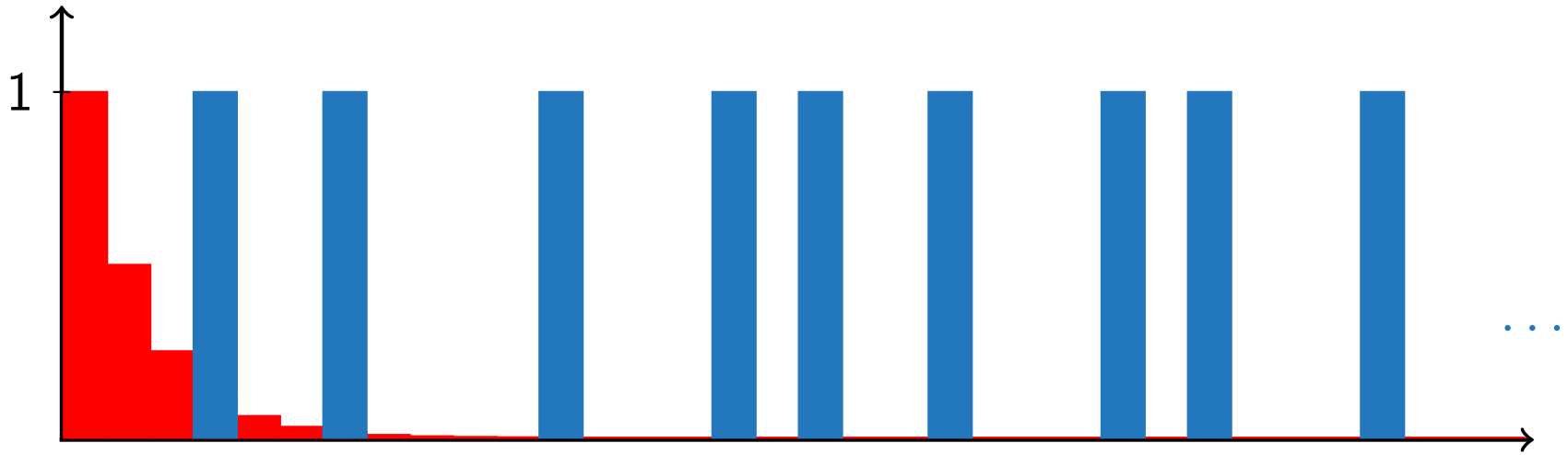

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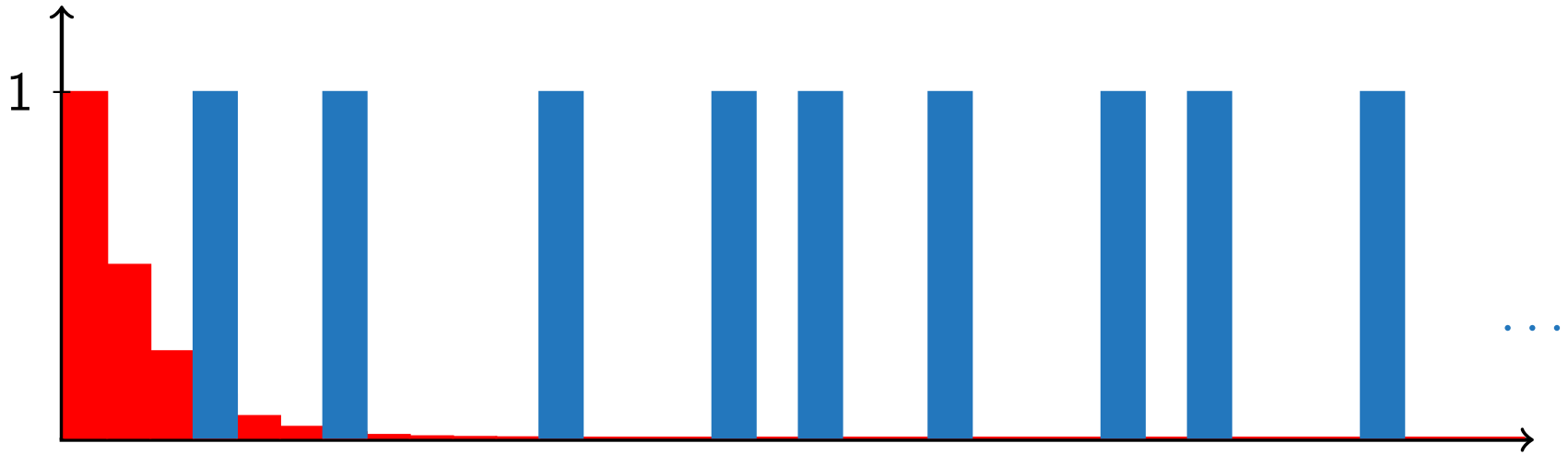
Case 2: $Q \notin \overline{\mathcal{UH}}$, i.e. Q terminates on every input.

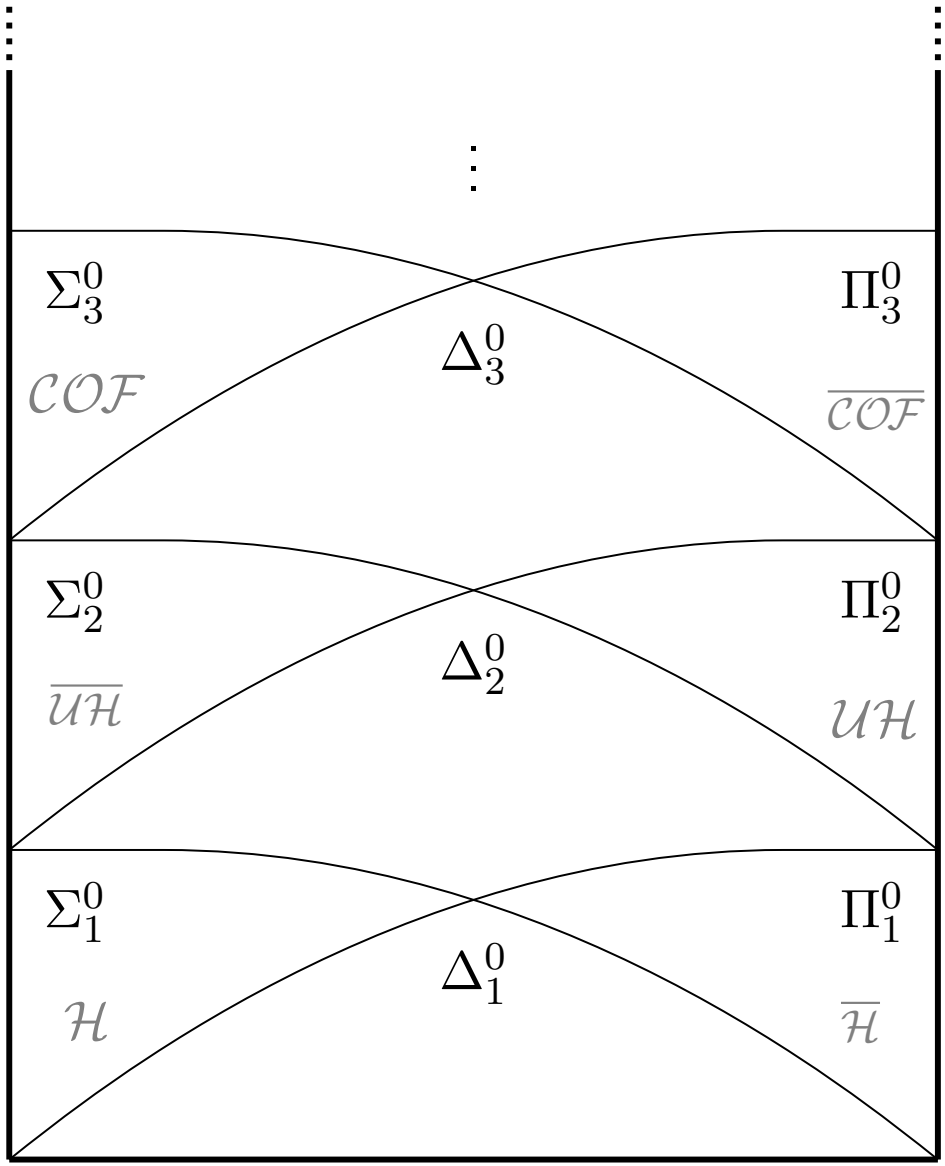
Expected runtime of P (integral over the bars):



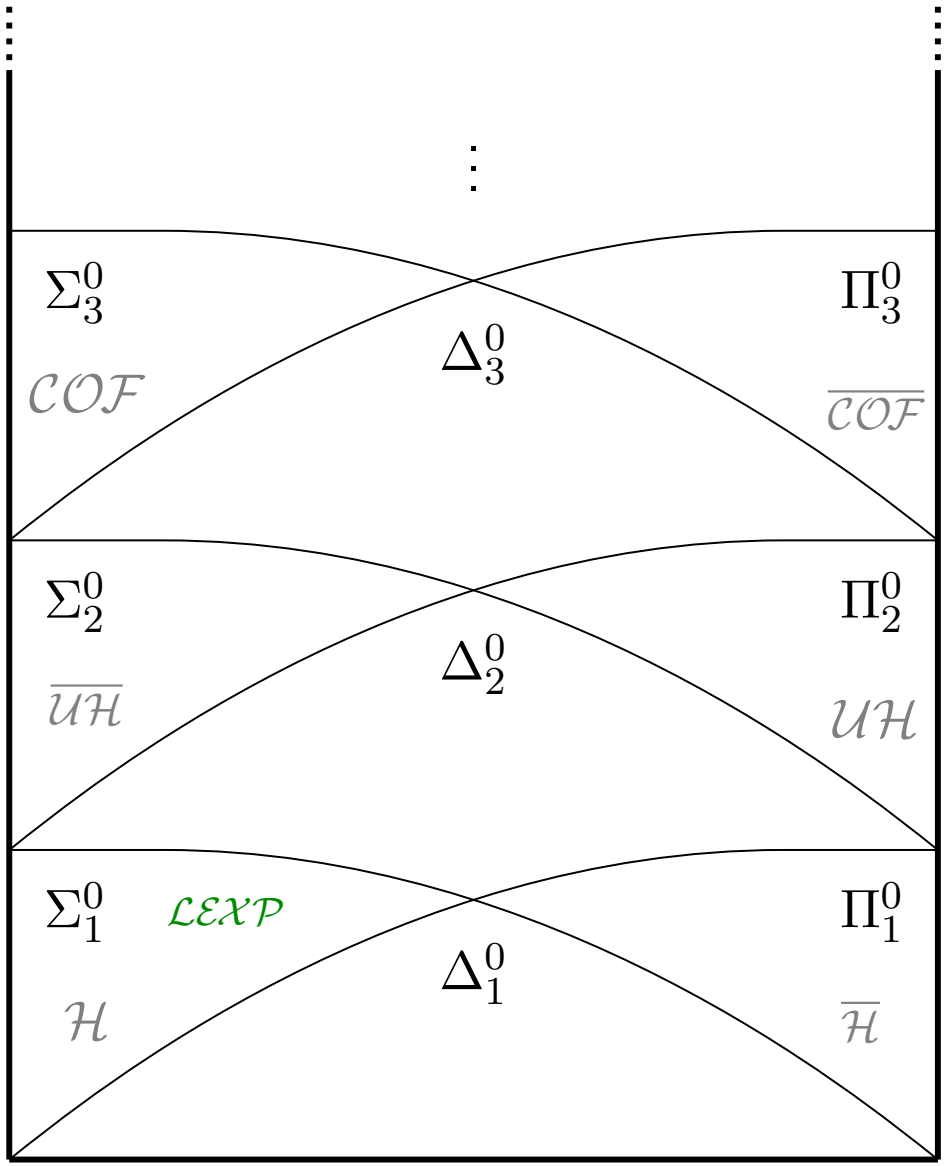
Infinite cheering — infinite expected runtime!

Summary

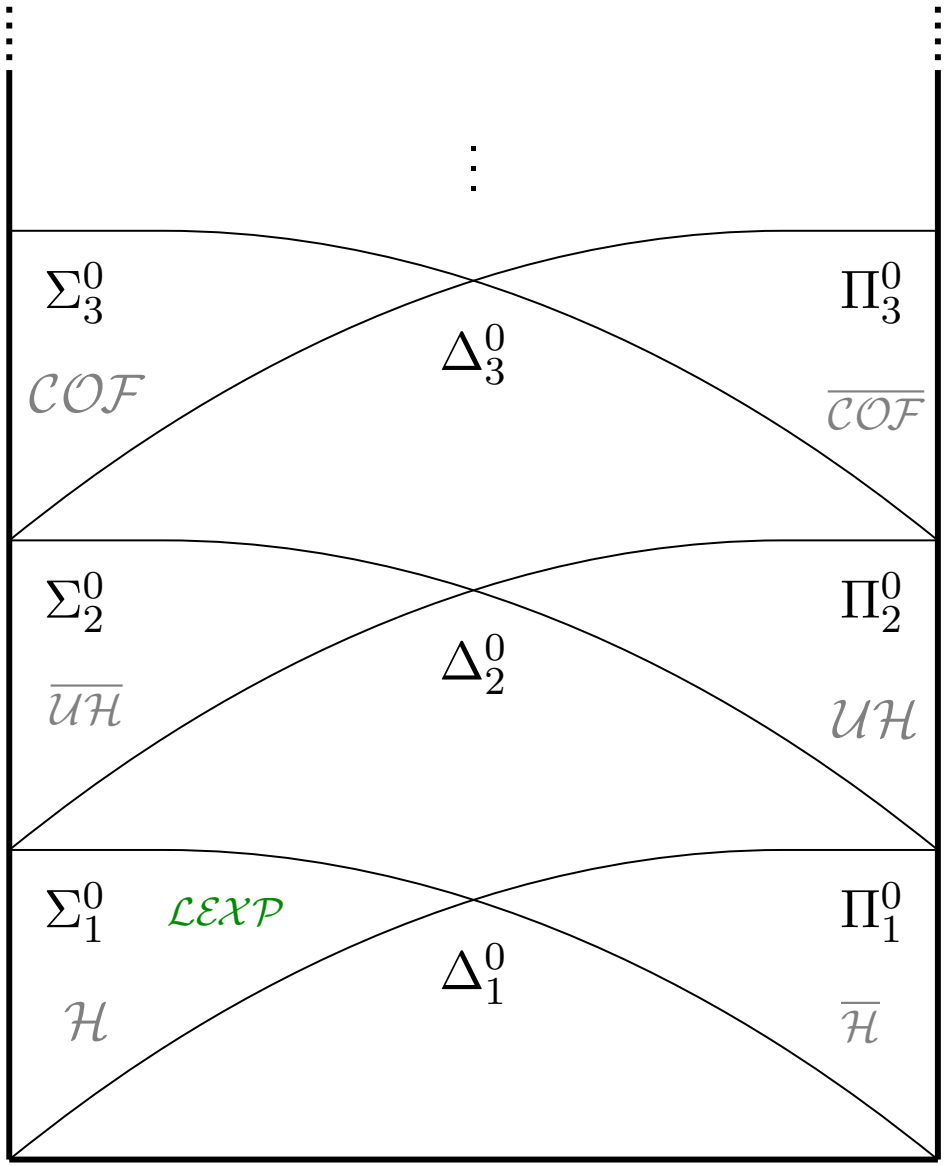
Summary


Σ_3^0
Δ_3^0
Π_3^0
$\mathcal{COF}$
$\overline{\mathcal{COF}}$
Σ_2^0
Δ_2^0
Π_2^0
$\overline{\mathcal{UH}}$
$\mathcal{UH}$
Σ_1^0
Δ_1^0
Π_1^0
$\mathcal{H}$
$\overline{\mathcal{H}}$

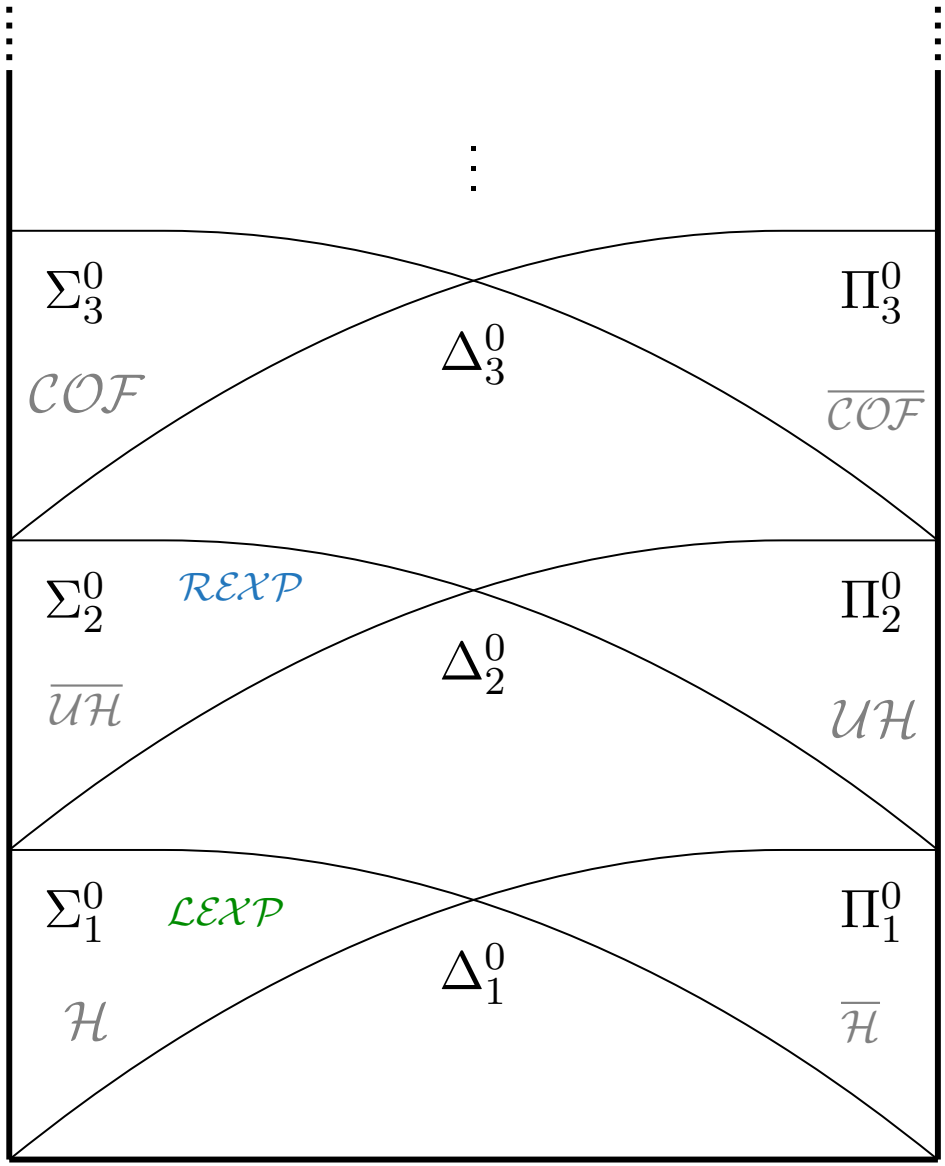
Summary


Σ_3^0
Π_3^0
Δ_3^0
$\mathcal{COF}$
$\overline{\mathcal{COF}}$
Σ_2^0
Π_2^0
Δ_2^0
$\overline{\mathcal{UH}}$
$\mathcal{UH}$
Σ_1^0
$\mathcal{LEXP}$
Π_1^0
Δ_1^0
$\mathcal{H}$
$\overline{\mathcal{H}}$

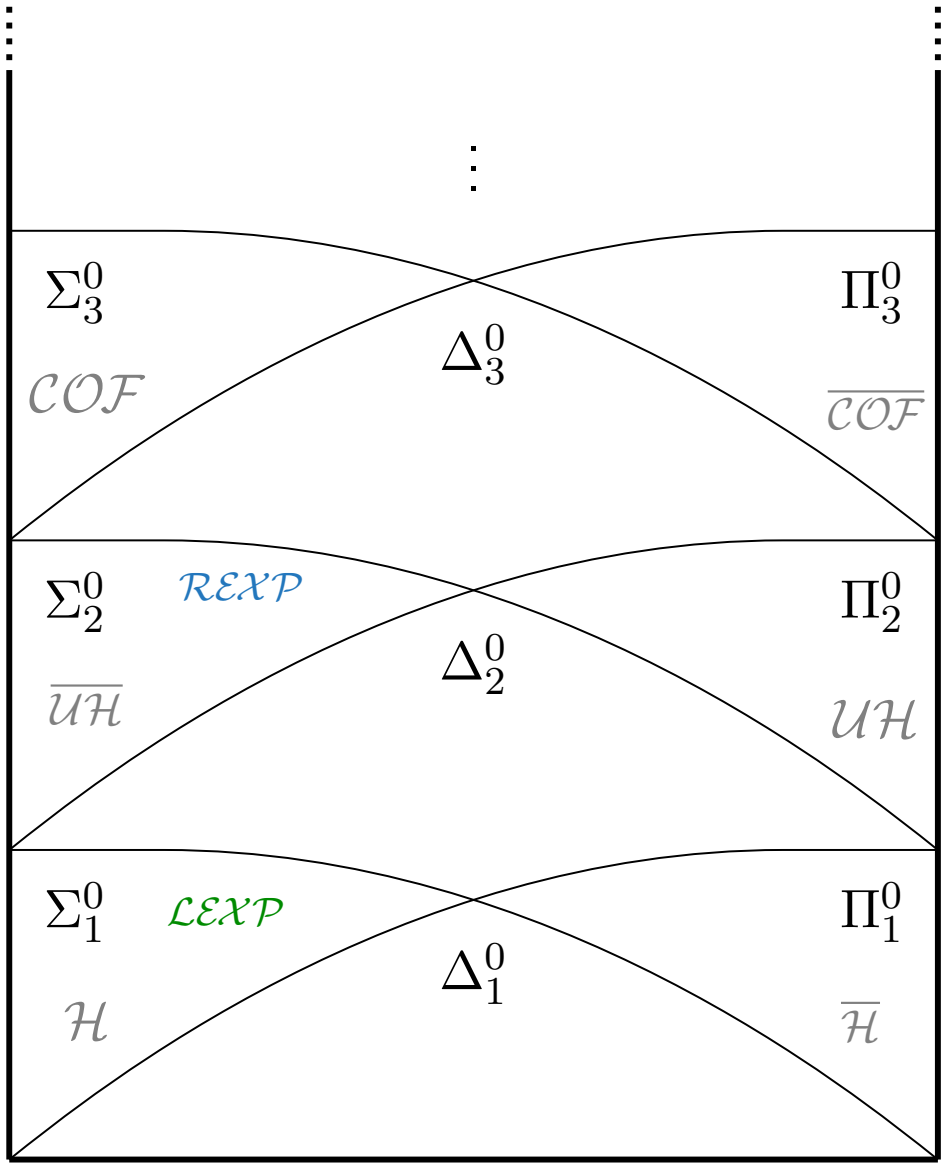
Summary


Σ_3^0
Π_3^0
Δ_3^0
$\mathcal{COF}$
$\overline{\mathcal{COF}}$
Σ_2^0
Π_2^0
Δ_2^0
$\overline{\mathcal{UH}}$
$\mathcal{UH}$
Σ_1^0
$\mathcal{LEXP}$
Π_1^0
Δ_1^0
$\mathcal{H}$
$\overline{\mathcal{H}}$
semi–decidable

Summary


Σ_3^0
Π_3^0
Δ_3^0
$\mathcal{COF}$
$\overline{\mathcal{COF}}$
Σ_2^0
$\mathcal{REXP}$
Π_2^0
Δ_2^0
$\overline{\mathcal{UH}}$
$\mathcal{UH}$
Σ_1^0
$\mathcal{LEXP}$
Π_1^0
Δ_1^0
semi–decidable
$\mathcal{H}$
$\overline{\mathcal{H}}$

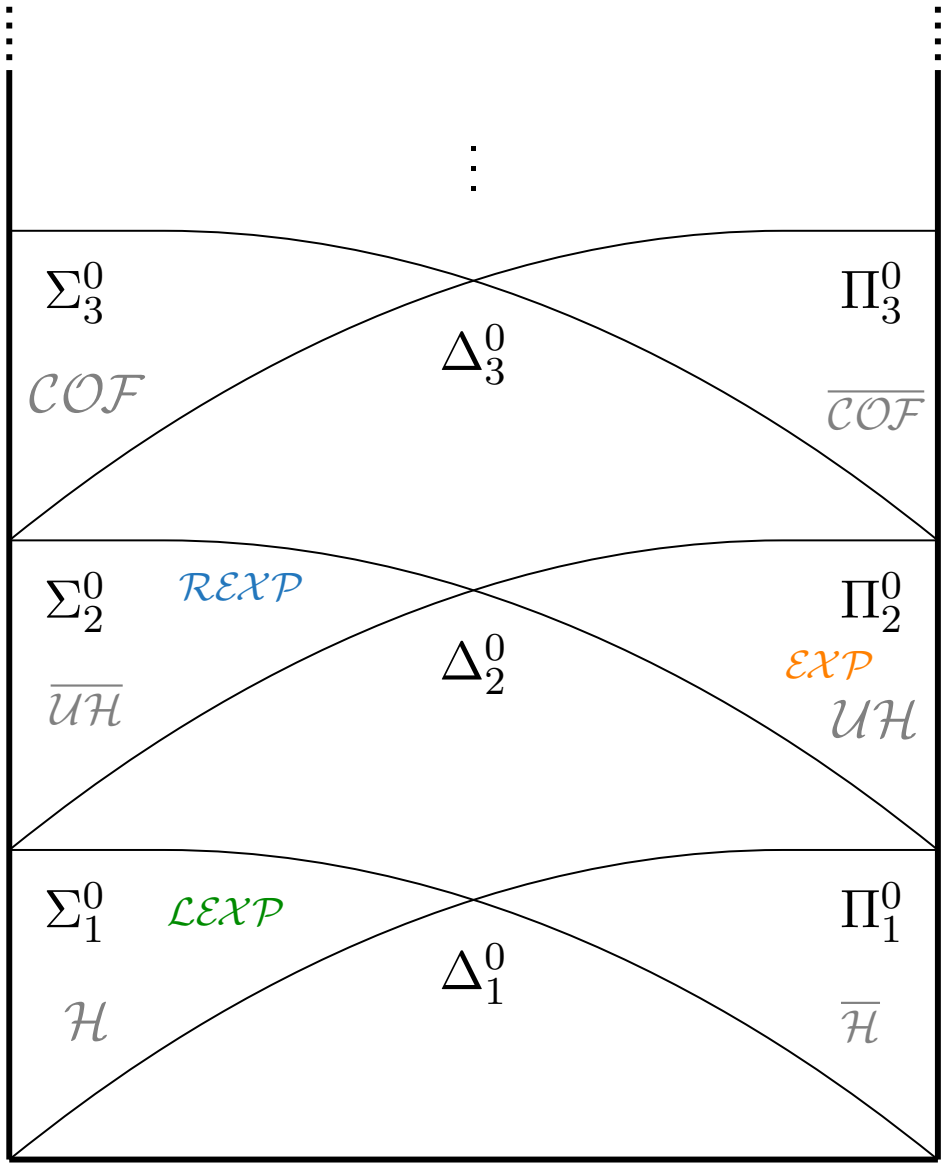
Summary



with access to $\mathcal{H}$–oracle: semi–decidable

semi–decidable

Summary



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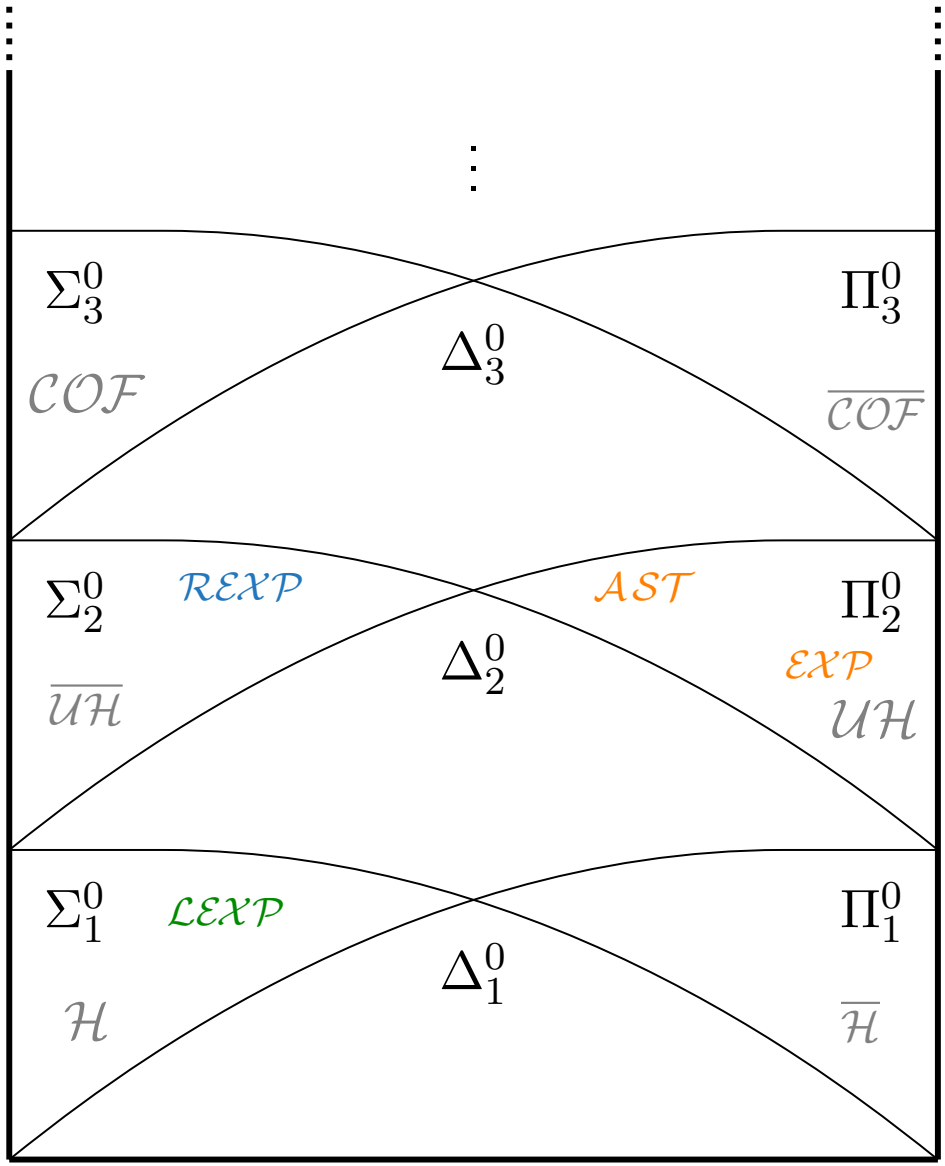
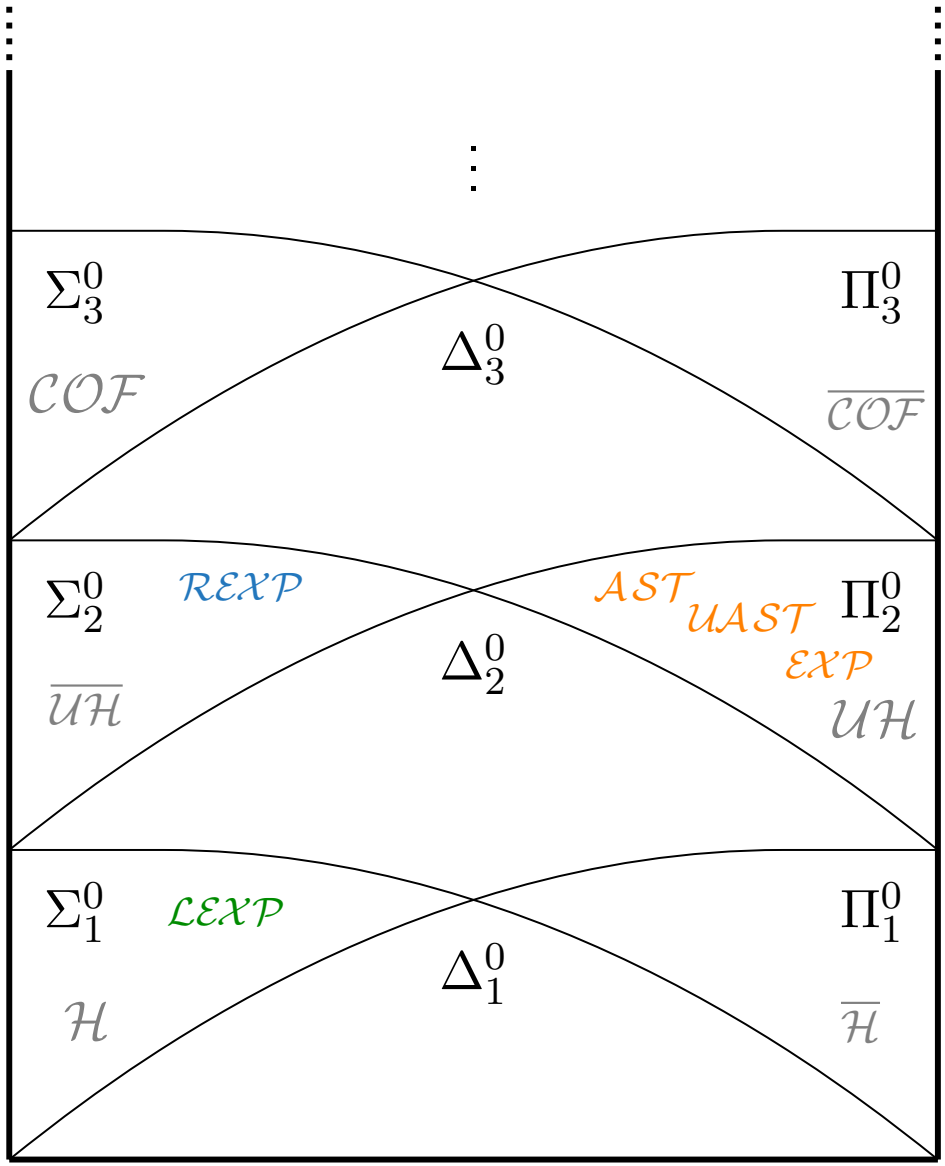
Summary



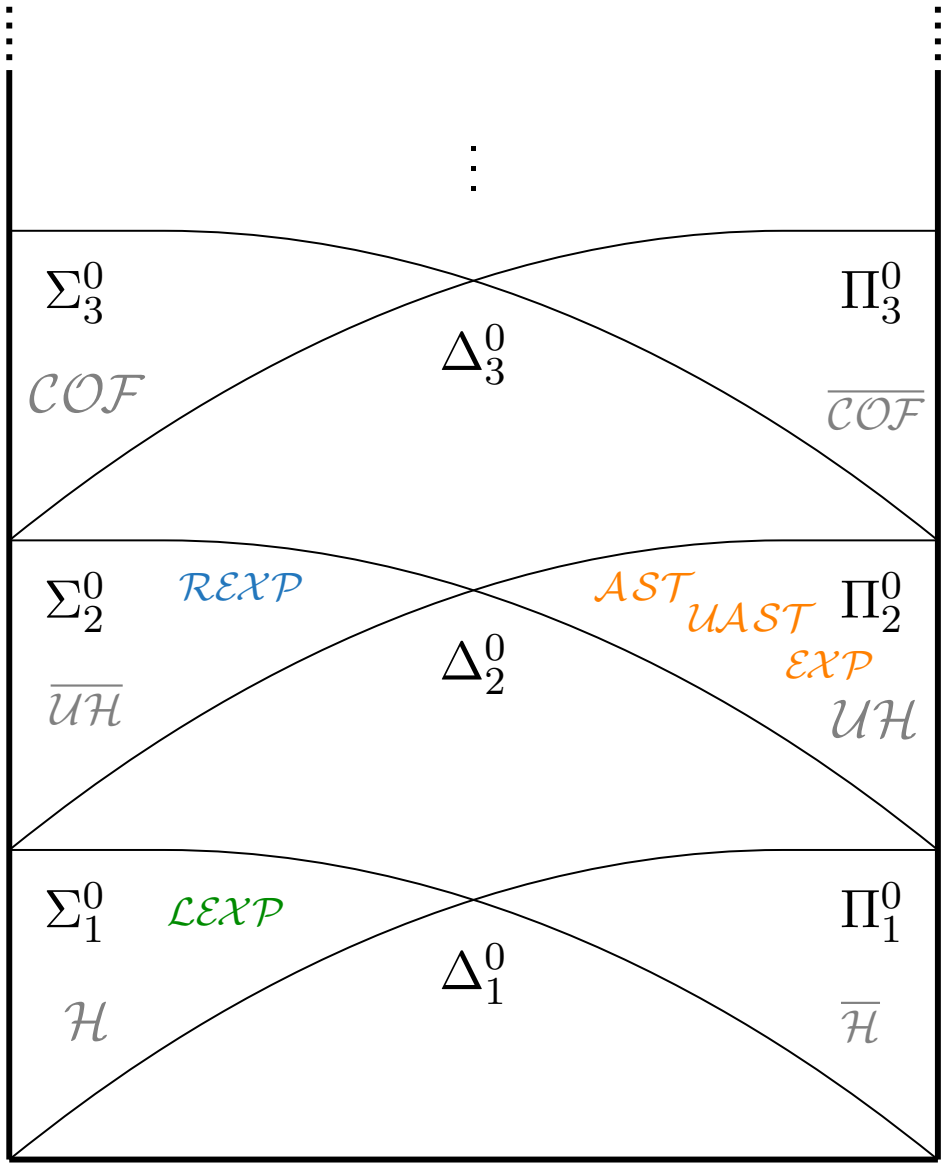
with access to $\mathcal{H}$–oracle: semi–decidable

semi–decidable

Summary


Σ_3^0
Π_3^0
Δ_3^0
$\mathcal{COF}$
$\overline{\mathcal{COF}}$
Σ_2^0
$\mathcal{REXP}$
$\mathcal{AST}$
$\mathcal{UAST}$
Π_2^0
Δ_2^0
$\mathcal{EXP}$
$\overline{\mathcal{UH}}$
$\mathcal{UH}$
with access to $\mathcal{H}$–oracle: semi–decidable
Σ_1^0
$\mathcal{LEXP}$
Π_1^0
Δ_1^0
semi–decidable
$\mathcal{H}$
$\overline{\mathcal{H}}$

Summary

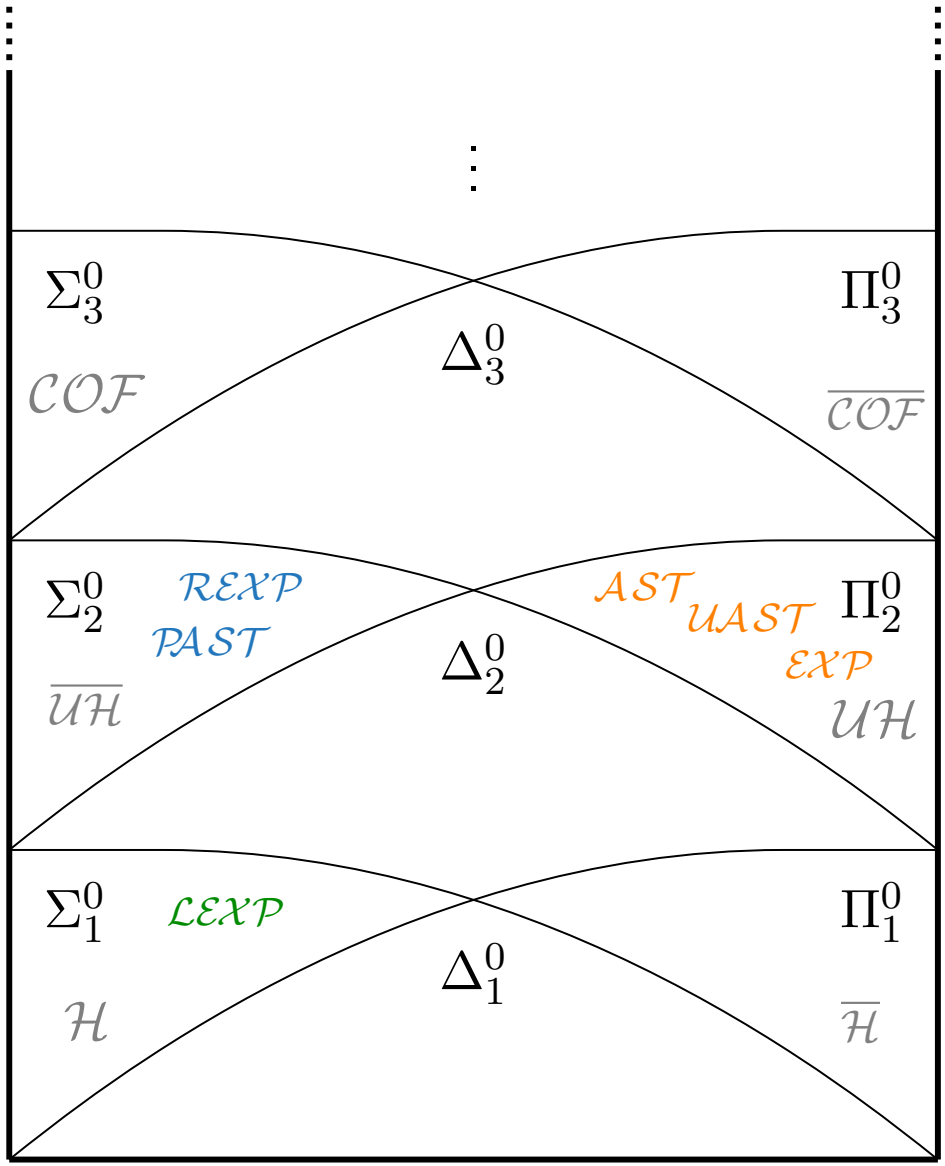


with access to $\mathcal{H}$–oracle: semi–decidable

semi–decidable

not semi–decidable; even with access to $\mathcal{H}$–oracle

Summary

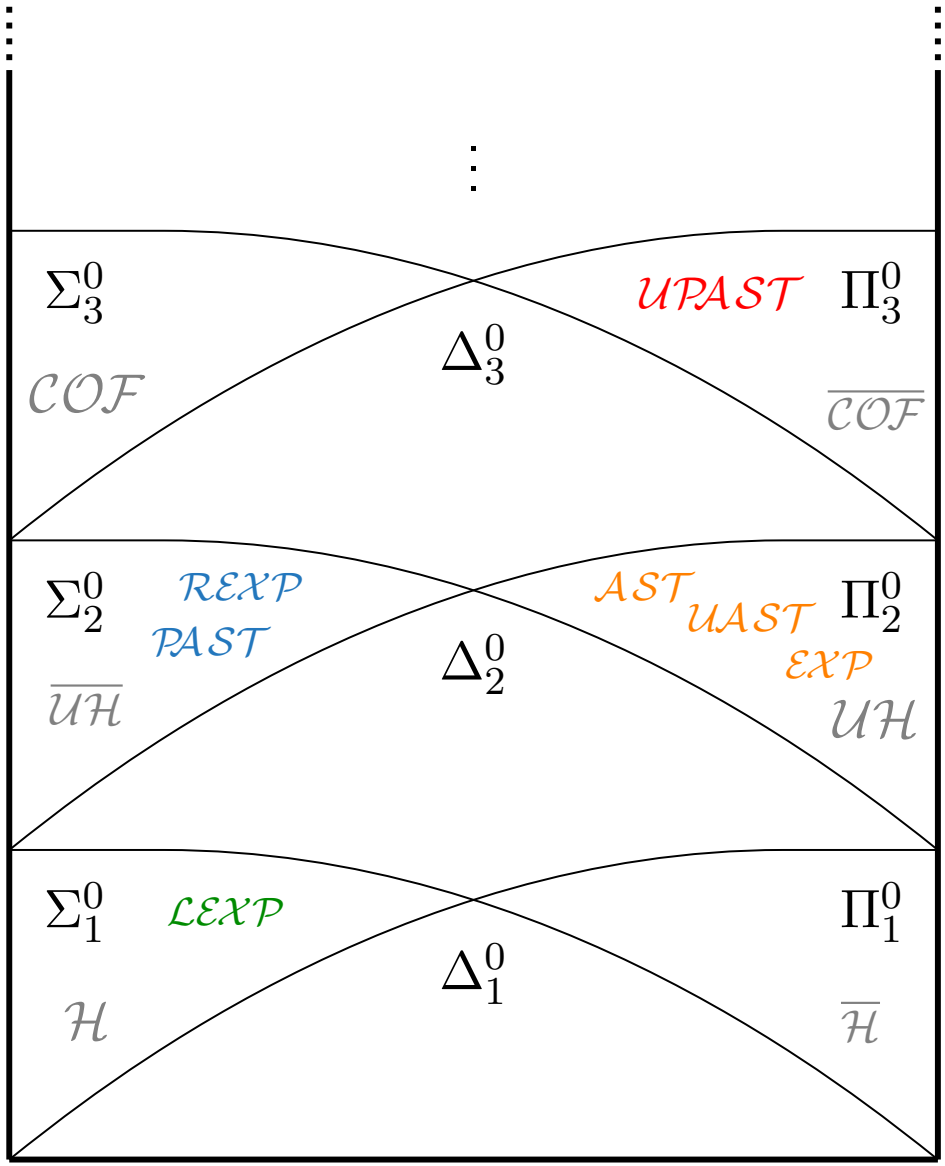


with access to $\mathcal{H}$–oracle: semi–decidable

semi–decidable

not semi–decidable; even with access to $\mathcal{H}$–oracle

Summary

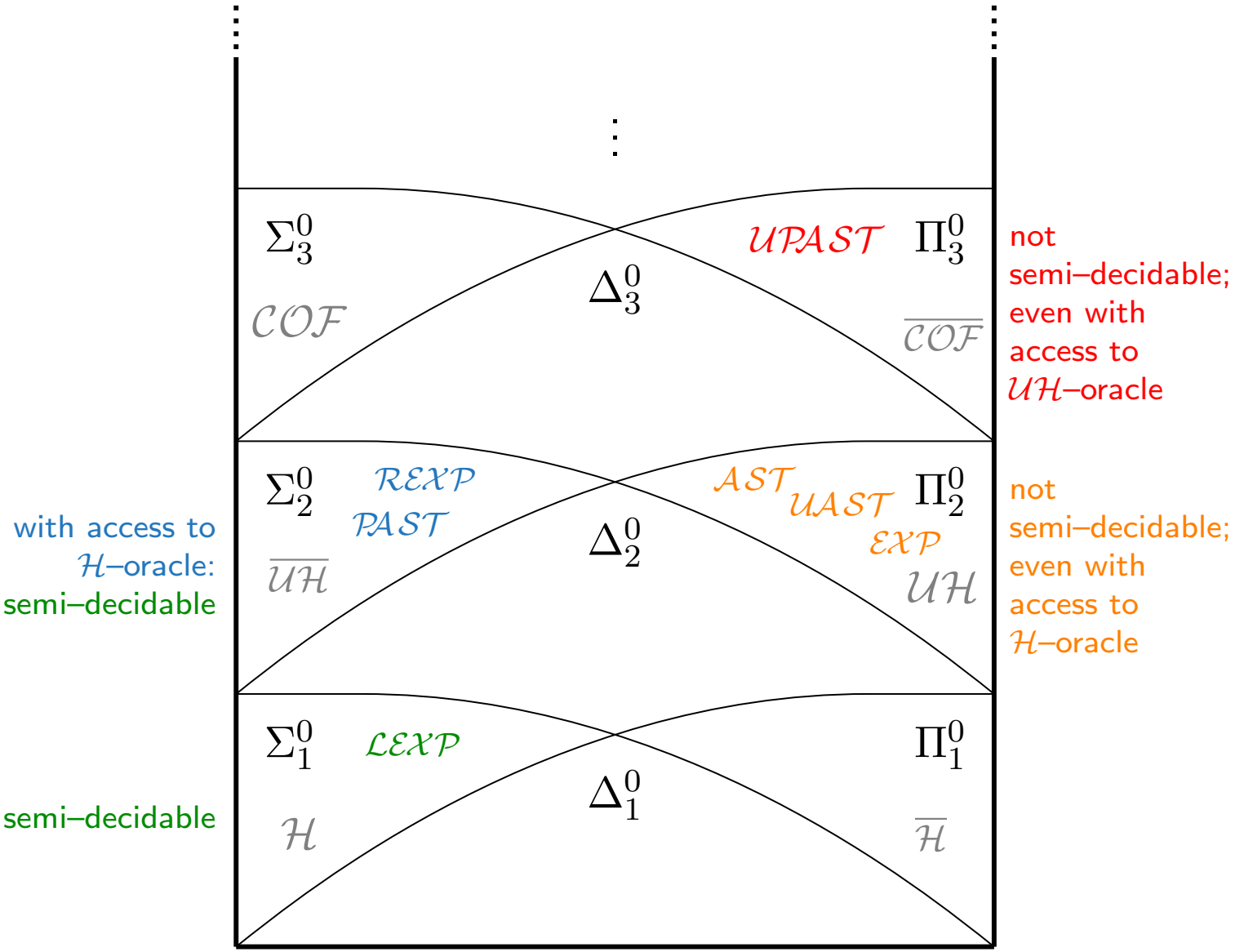


with access to $\mathcal{H}$–oracle: semi–decidable

semi–decidable

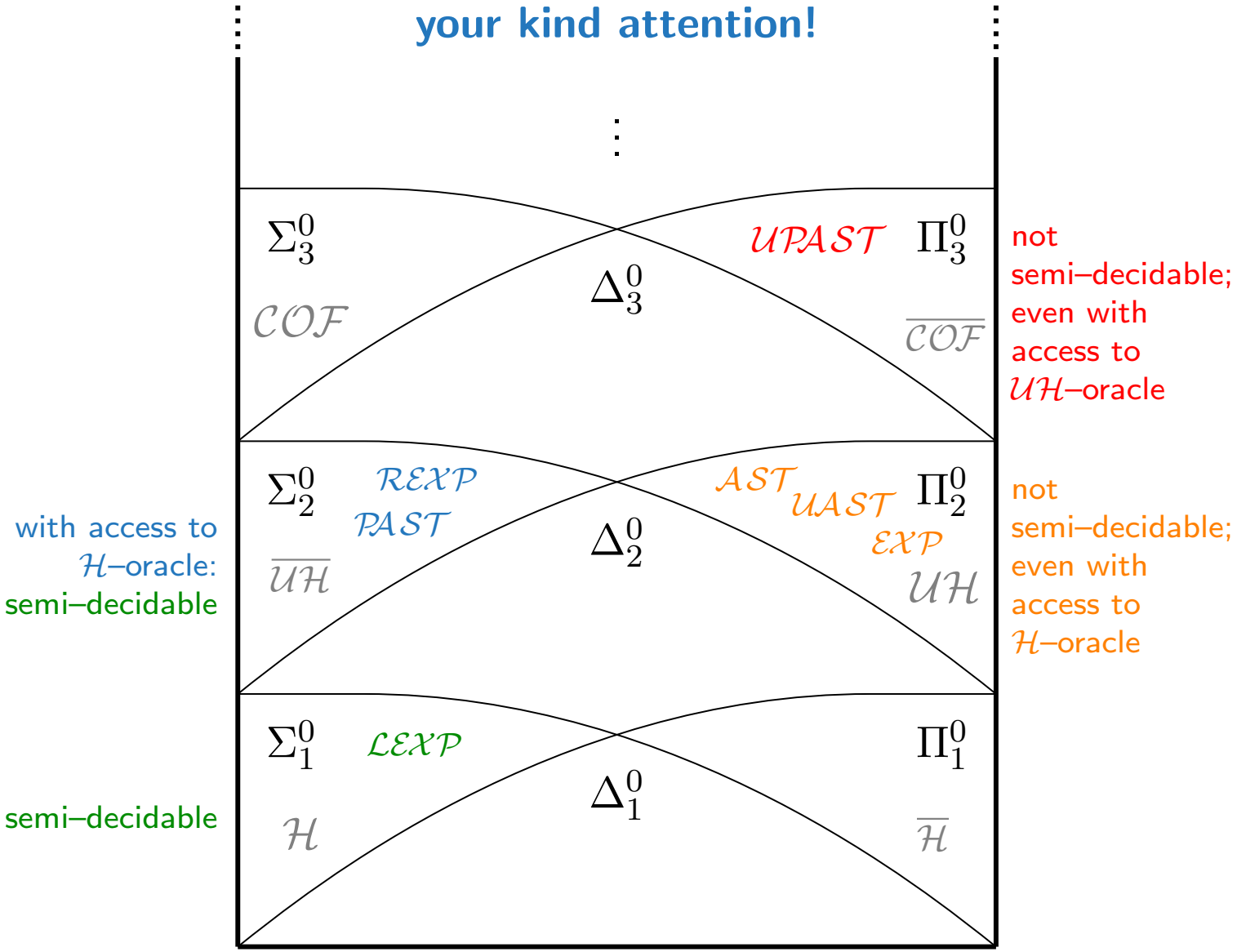
not semi–decidable; even with access to $\mathcal{H}$–oracle

Summary


Σ_3^0
$\mathcal{UPAST}$
Π_3^0
Δ_3^0
$\mathcal{COF}$
$\overline{\mathcal{COF}}$
not semi–decidable; even with access to $\mathcal{UH}$–oracle
Σ_2^0
$\mathcal{REXP}$
$\mathcal{PAST}$
$\mathcal{AST}$
$\mathcal{UAST}$
Π_2^0
$\mathcal{EXP}$
Δ_2^0
$\overline{\mathcal{UH}}$
$\mathcal{UH}$
with access to $\mathcal{H}$–oracle: semi–decidable
not semi–decidable; even with access to $\mathcal{H}$–oracle
Σ_1^0
$\mathcal{LEXP}$
Π_1^0
Δ_1^0
$\mathcal{H}$
$\overline{\mathcal{H}}$
semi–decidable

Summary

Thank you for your kind attention!



Proof of $\mathcal{LEXP} \in \Sigma_1^0$

$\mathcal{LEXP}$ is defined by the following formula:

$$\exists y \colon q < \sum_{k=0}^{y} \sum_{w \in \{L,\, R\}^{\leq k}} \wp\Big(\mathsf{T}_k\big(\sigma_{P,\eta},\, w\big),\, v\Big)$$

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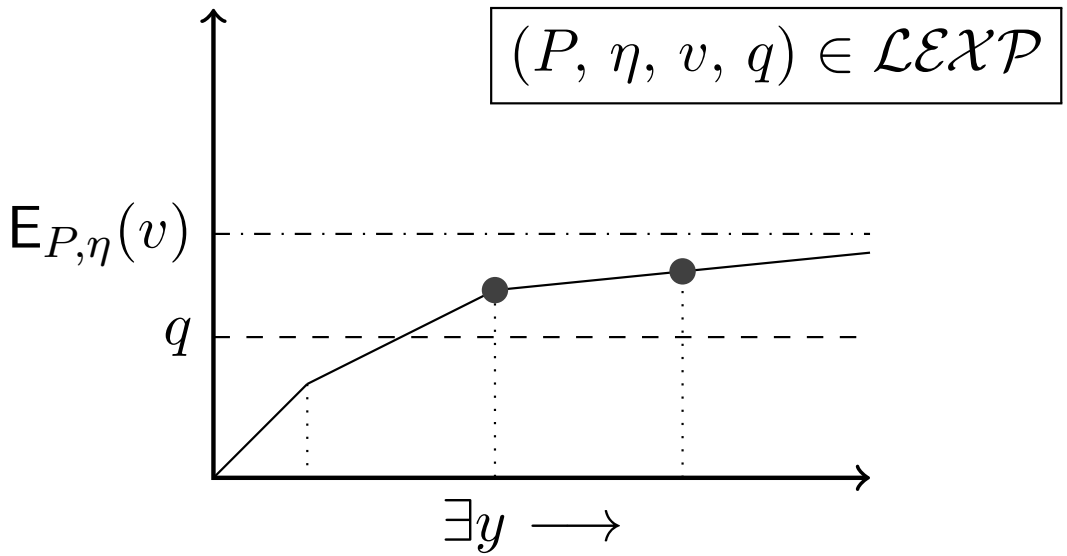

Hardness of Deciding Universal Positive A.–S. Termination

Hardness of Deciding Universal Positive A.–S. Termination

$\overline{\mathcal{COF}}$ — A Very Hard Problem

The problem $\mathcal{COF}$, defined by

$$\mathcal{COF} \;=\; \left\{ Q \in \text{ordProg} \;\middle|\; \{\eta \mid Q \text{ halts on input } \eta\} \text{ is cofinite} \right\},$$

is Σ_3^0–complete.

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is Σ^0_3–complete. The complement $\overline{\mathcal{COF}}$ is Π^0_3–complete.

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Theorem

$\mathcal{UPAST}$ is Π_3^0–complete.

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Theorem

$\mathcal{UPAST}$ is Π_3^0–complete.

Proof: $\mathcal{UPAST} \in \Pi_3^0$ as $\mathcal{PAST} \in \Sigma_2^0$ and $\mathcal{UPAST}$ is in universal closure of $\mathcal{PAST}$.

Hardness of Deciding Universal Positive A.–S. Termination

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The problem $\mathcal{COF}$, defined by

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Theorem

$\mathcal{UPAST}$ is Π_3^0–complete.

Proof: $\mathcal{UPAST} \in \Pi_3^0$ as $\mathcal{PAST} \in \Sigma_2^0$ and $\mathcal{UPAST}$ is in universal closure of $\mathcal{PAST}$. $\mathcal{UPAST}$ is Π_3^0–hard, as $\overline{\mathcal{COF}} \leq_{\mathsf{m}} \mathcal{UPAST}$.