Analyzing Expected Outcomes and Almost–Sure Termination of Probabilistic Programs is Hard

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How hard is it to solve these analysis problems?

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1. The arithmetical hierarchy
2. Probabilistic programs
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   - Semantics
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4. Hardness of deciding almost–sure termination

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- Class Σ_n^0 is defined as

$$\Sigma_n^0 = \Big\{ A \;\Big|\; A = \{\vec{x} \mid \exists y_1 \, \forall y_2 \, \exists y_3 \, \cdots \exists/\forall y_n : (\vec{x}, y_1, y_2, y_3, \ldots, y_n) \in R\}, R \text{ is a decidable relation} \Big\}$$

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- n is really about the number of alternating quantifiers

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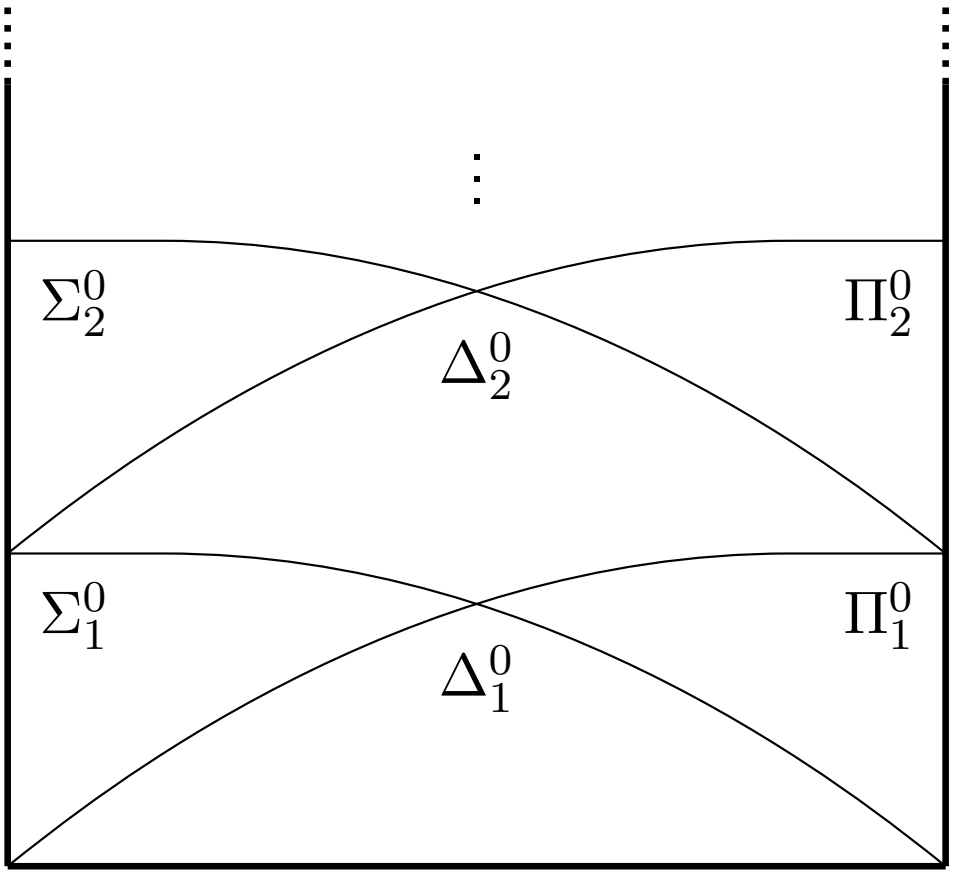

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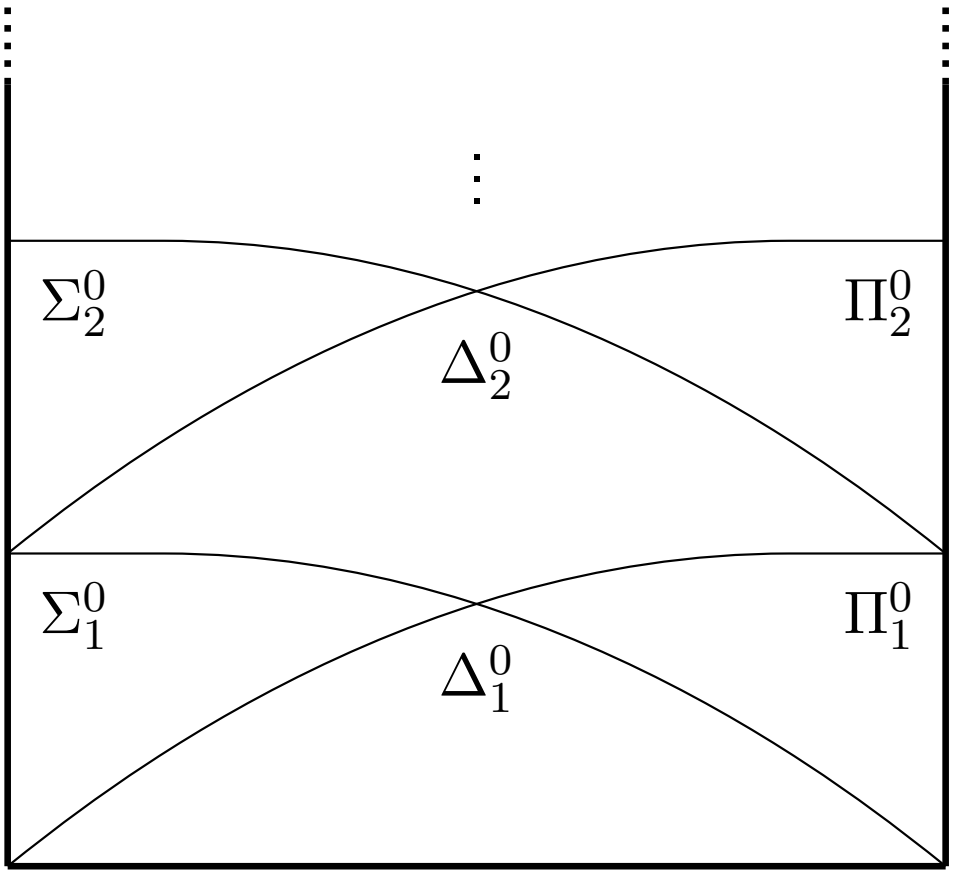

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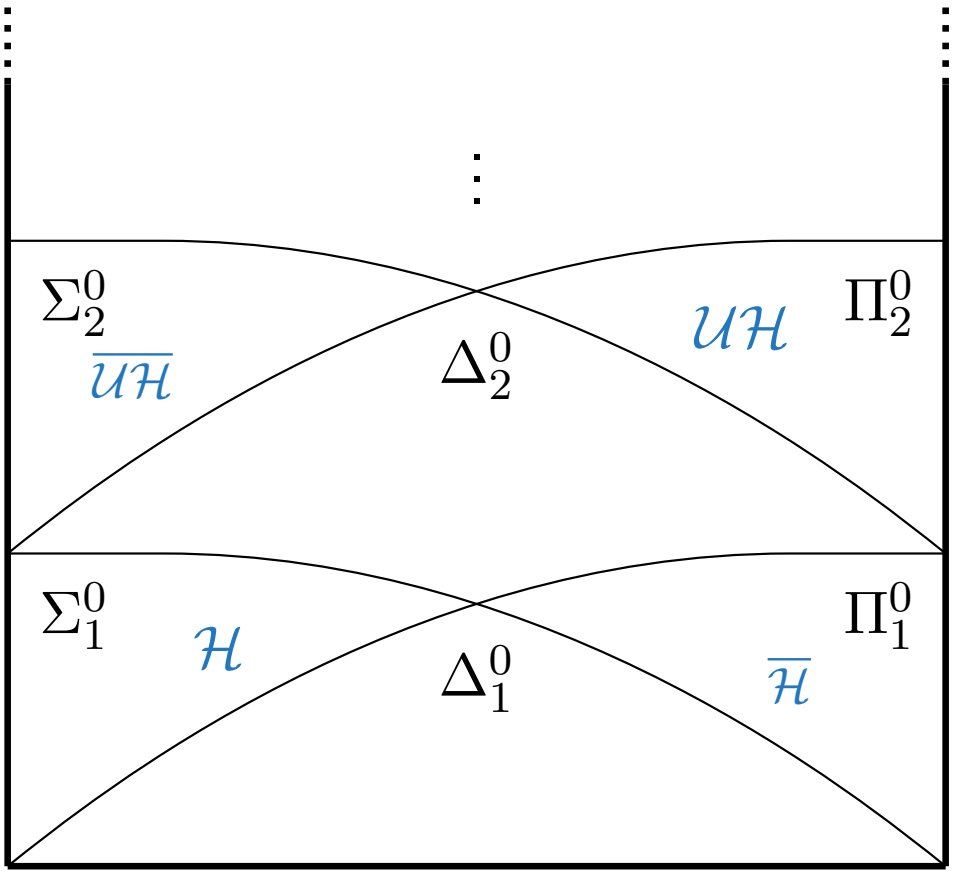

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- Use $\sigma \vdash^k \tau$ in the usual sense
- Write $\sigma \vdash_{\text{(name)}} \tau$ if τ is inferred by the use of the (name)–rule (for name $\in \{\text{assign, concat1, } \ldots\}$)

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How do we deal with probabilistic choice?

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$$\mathsf{T}(\sigma) = \begin{cases} \tau, & \text{if } \sigma \vdash \tau \\ \top, & \text{else (i.e. if } \sigma = \langle \downarrow, \eta, a, \theta \rangle) \end{cases}$$

How do we deal with probabilistic choice?

Basically, we simply tell T how to resolve it!

State Successors — The Probabilistic Case

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$$\mathsf{T}_{prob}(\sigma, s) = \begin{cases} \tau_L, & \text{if } s = L \text{ and } \sigma \vdash_{\text{(prob1)}} \tau_L \\ \tau_R, & \text{if } s = R \text{ and } \sigma \vdash_{\text{(prob2)}} \tau_R \end{cases}$$

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Provided s, the successor of σ is unique!

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- Otherwise, T^*_{prob} returns $\top$

Some Helper Functions

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Extracting Probabilities and Variable Values of Terminal States

$$\alpha(\sigma) = \begin{cases} a, & \text{if } \sigma = \langle \downarrow, \eta, a, \theta \rangle \\ 0, & \text{otherwise} \end{cases}$$

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Computable Enumeration of all $w \in \{L, R\}^*$

There exists a computable bijection $h \colon \mathbb{N} \to \{L, R\}^*$.

Expected Outcomes

The Expected Outcome $\mathsf{E}_P(v)$

The expected outcome of variable v after executing program P:

$$\mathsf{E}_P(v) \;:=\; \sum_{i\in\mathbb{N}} \sum_{k\in\mathbb{N}} \wp\Big(\mathsf{T}^*_{prob}\big(\langle P,\, \eta_0,\, 1,\, \varepsilon\rangle,\, k,\, h(i)\big),\, v\Big)$$

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- For each terminal state, take the probability times the value of v as the summand

Termination Probabilities

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The probability that P terminates:

$$\mathsf{Pr}_P(\downarrow) \;:=\; \sum_{i \in \mathbb{N}} \sum_{k \in \mathbb{N}} \alpha\Big(\mathsf{T}^*_{prob}\big(\langle P,\, \eta_0,\, 1,\, \varepsilon\rangle,\, k,\, h(i)\big)\Big)$$

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Decision Problems We Analyzed

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Lower and Upper Bounds, and Exact Expected Outcomes

$$\begin{aligned}(P, v, q) \in \mathcal{LEXP} &:\Longleftrightarrow q < \mathsf{E}_P(v)\\ (P, v, q) \in \mathcal{UEXP} &:\Longleftrightarrow q > \mathsf{E}_P(v)\\ (P, v, q) \in \mathcal{EXP} &:\Longleftrightarrow q = \mathsf{E}_P(v)\end{aligned}$$

Hardness of Deciding $\mathcal{LEXP}$

Recursive Enumerability of $\mathcal{LEXP}$

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Lemma

$\mathcal{LEXP} \in \Sigma_1^0$, thus $\mathcal{LEXP}$ is recursively enumerable

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Proof of $\mathcal{LEXP} \in \Sigma_1^0$

$\mathcal{LEXP}$ is defined by the following formula:

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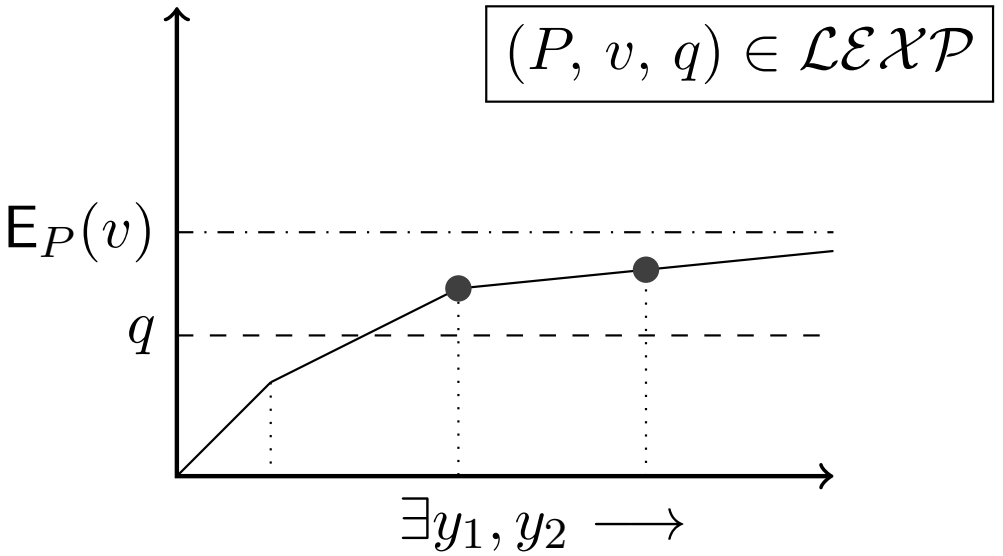

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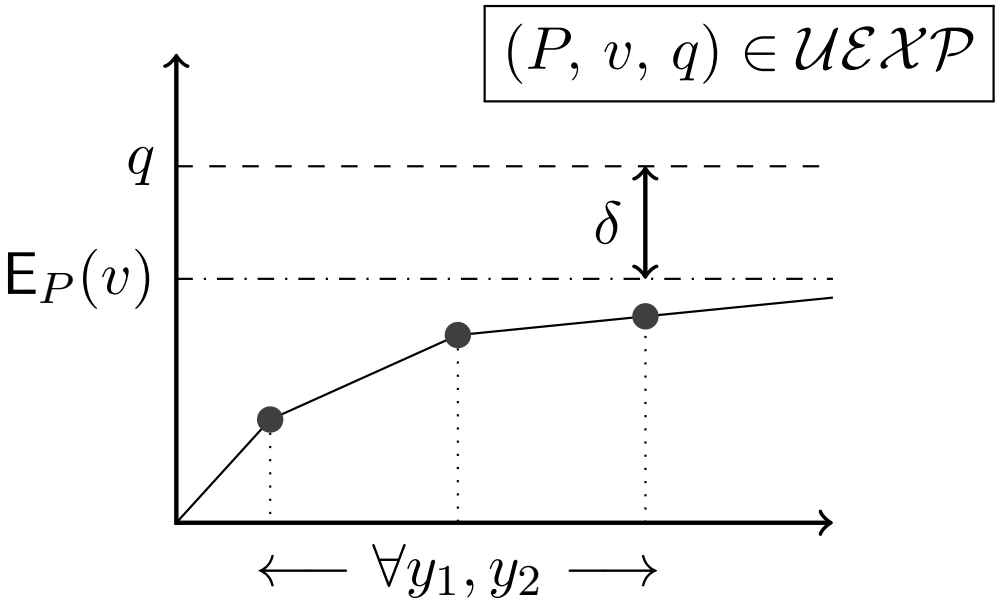

Actual Hardness of Deciding $\mathcal{UEXP}$ and $\mathcal{EXP}$

The Universal Halting Problem $\mathcal{UH}$

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Completenesses of $\mathcal{UH}$ and $\overline{\mathcal{UH}}$

$\mathcal{UH}$ is Π_2^0–complete and $\overline{\mathcal{UH}}$ is Σ_2^0–complete.

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First observation

There is a computable enumeration $g_Q \colon \mathbb{N} \to \{\mathsf{Var} \to \mathbb{Q}\}$ enumerating all possible variable valuations subject to

$$\forall i \in \mathbb{N}\ \forall v \in \mathsf{Var}\colon \Big[\big[g_Q(i)\big](v) \neq 0\Big] \implies \Big[v \text{ occurs in } Q\Big]$$

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Candidate for f

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where TQ is a program that simulates Q on input $g_Q(i)$.

Partial Correctness

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Probabilistic Program P Returned by $f(Q)$

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- While–loop establishes geometric distribution on i — hence, by $g_Q(i)$, a geometric distribution on all possible inputs for Q
- Then program P terminates with probability $\sum_{k\in\mathbb{N}} \frac{1}{2^k} = 1$ iff the simulation of Q terminates on every possible input $g_Q(i)$

Total Correctness

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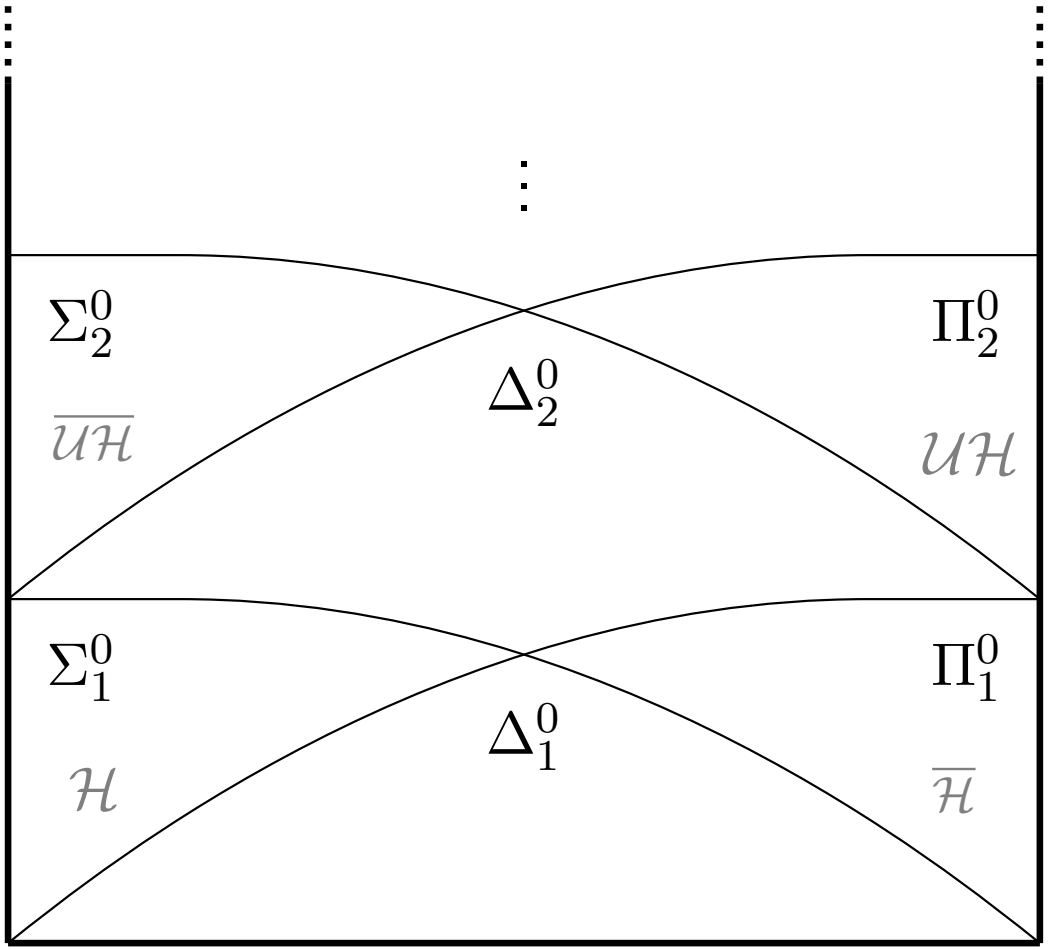
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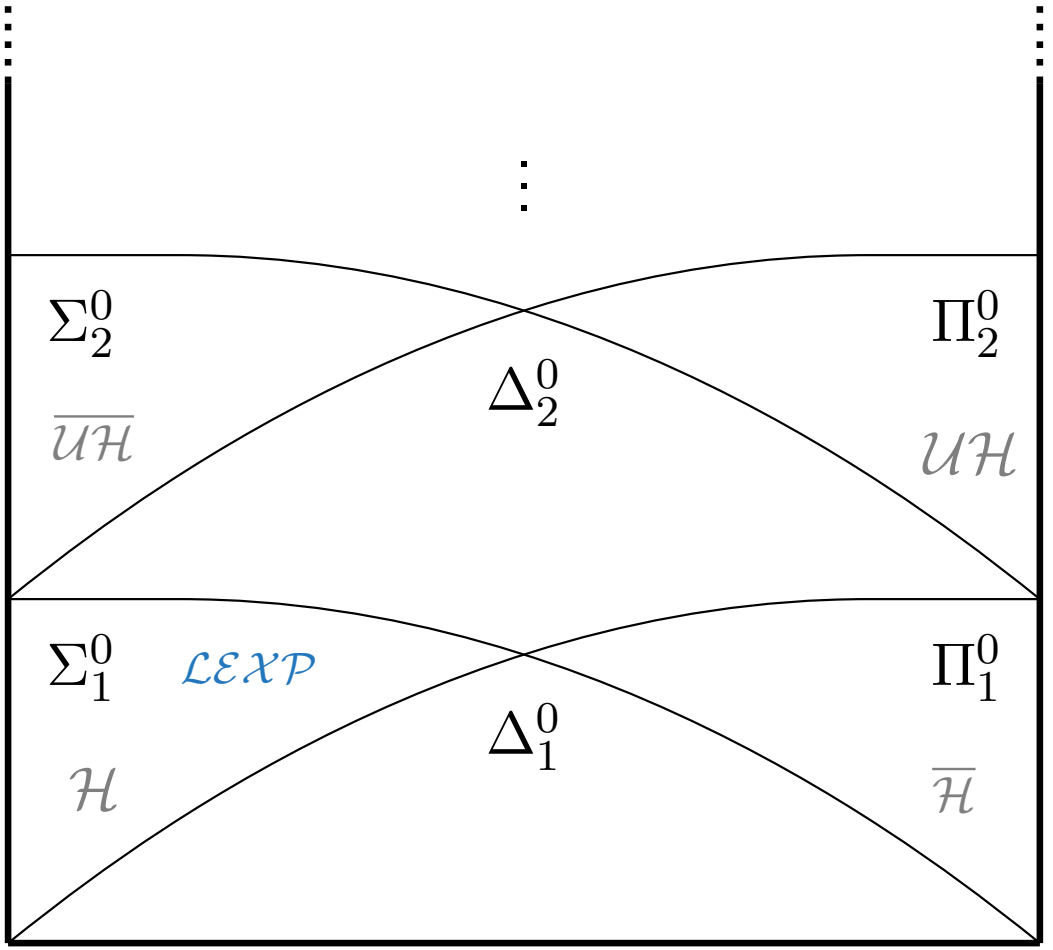
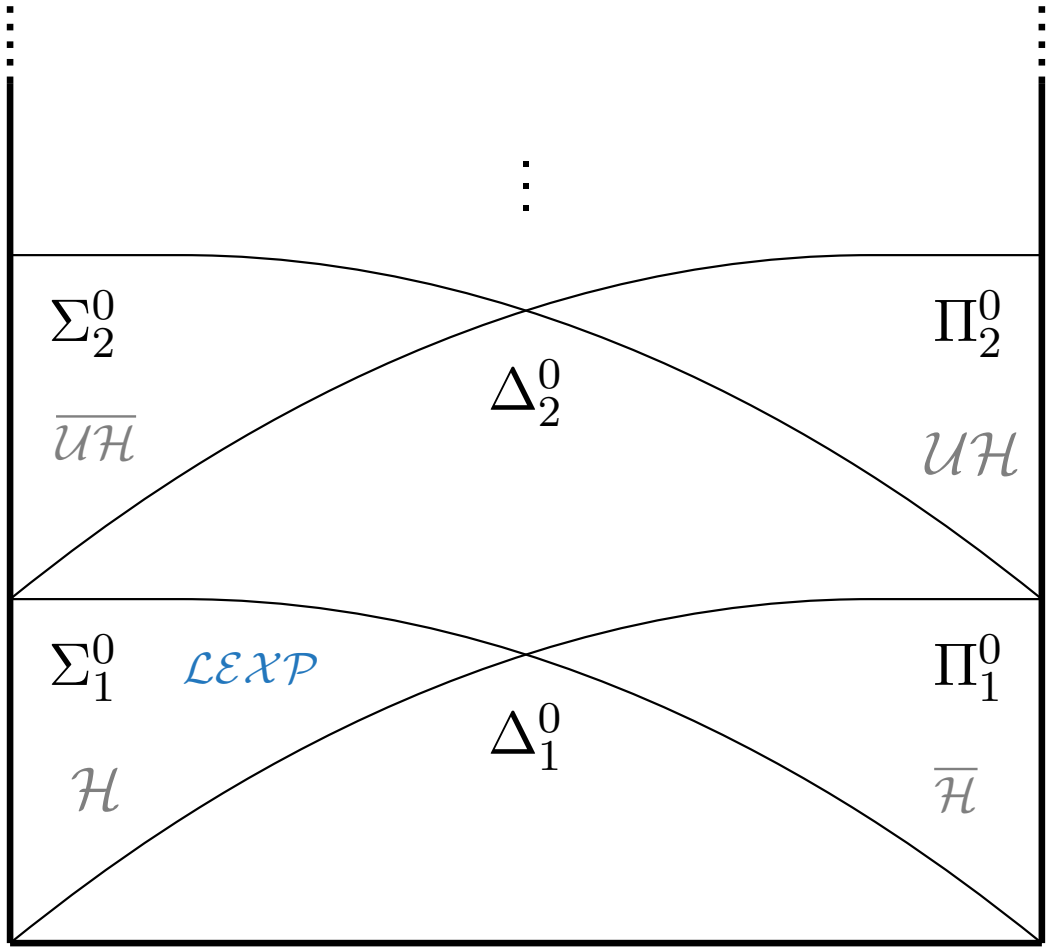
$\overline{\mathcal{H}}$

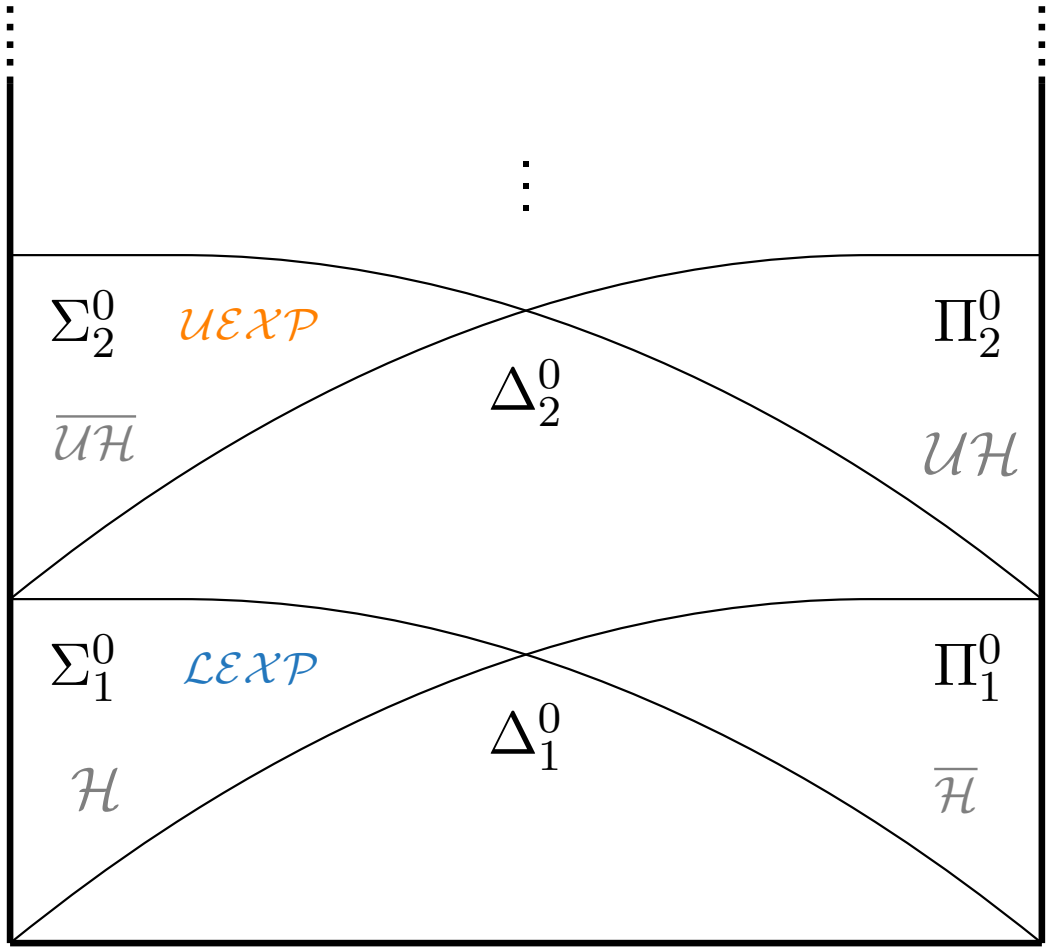
Summary


Σ_2^0
Π_2^0
Δ_2^0
$\overline{\mathcal{UH}}$
$\mathcal{UH}$
Σ_1^0
$\mathcal{LEXP}$
Π_1^0
Δ_1^0
$\mathcal{H}$
$\overline{\mathcal{H}}$

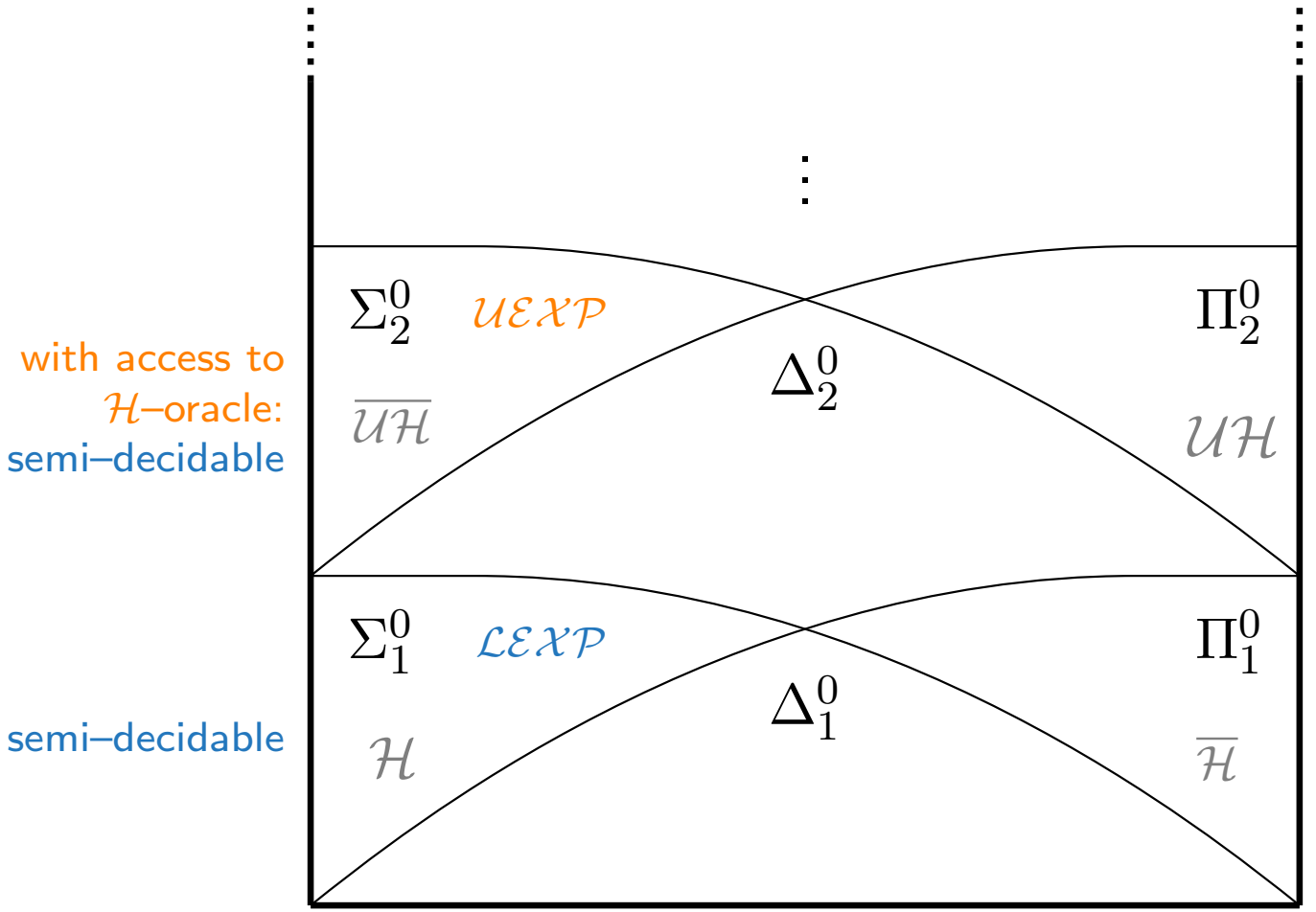
Summary


Σ_2^0
Π_2^0
Δ_2^0
$\overline{\mathcal{UH}}$
$\mathcal{UH}$
Σ_1^0
$\mathcal{LEXP}$
Π_1^0
Δ_1^0
semi–decidable
$\mathcal{H}$
$\overline{\mathcal{H}}$

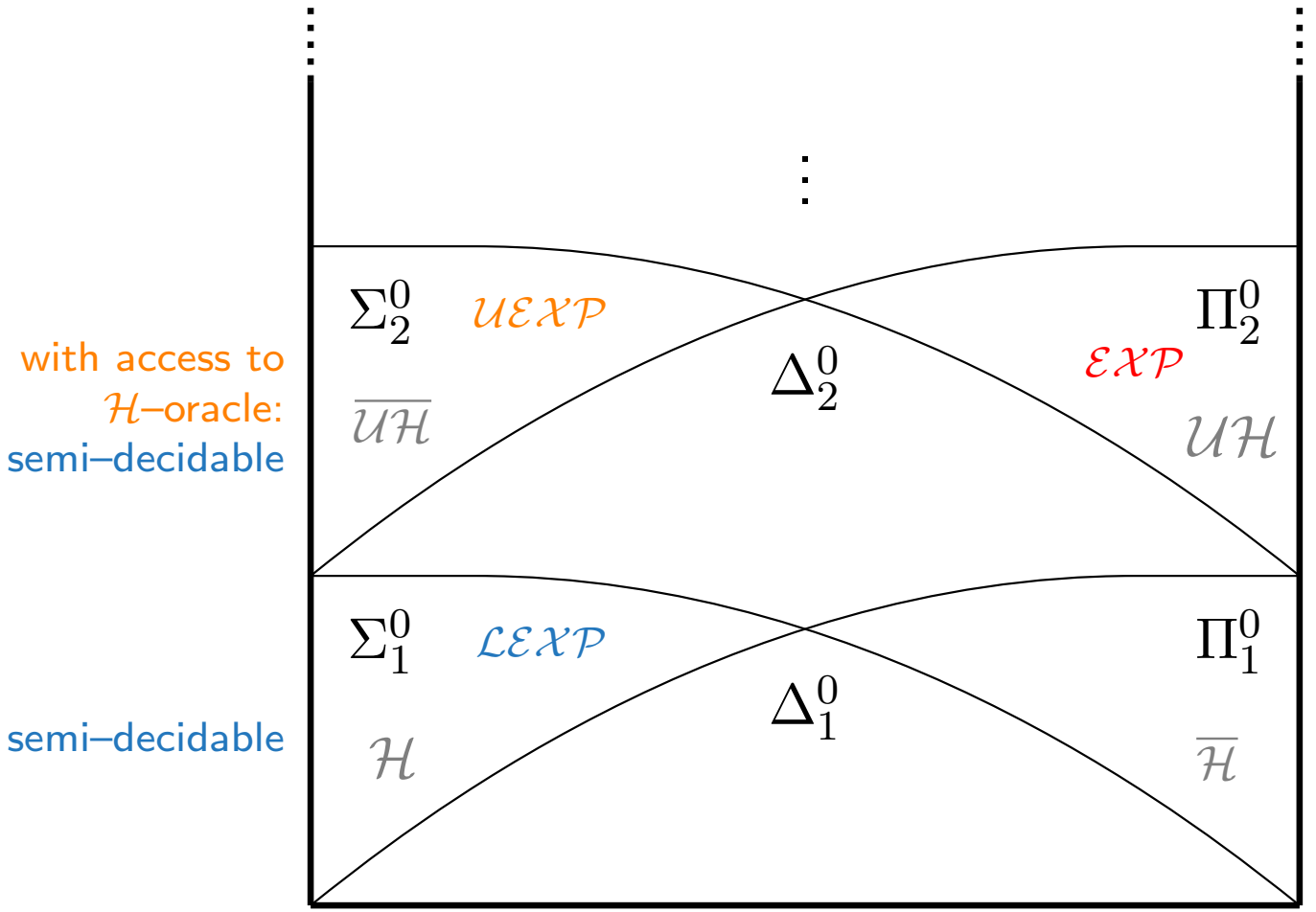
Summary


Σ_2^0
$\mathcal{UEXP}$
Π_2^0
Δ_2^0
$\overline{\mathcal{UH}}$
$\mathcal{UH}$
Σ_1^0
$\mathcal{LEXP}$
Π_1^0
Δ_1^0
semi–decidable
$\mathcal{H}$
$\overline{\mathcal{H}}$

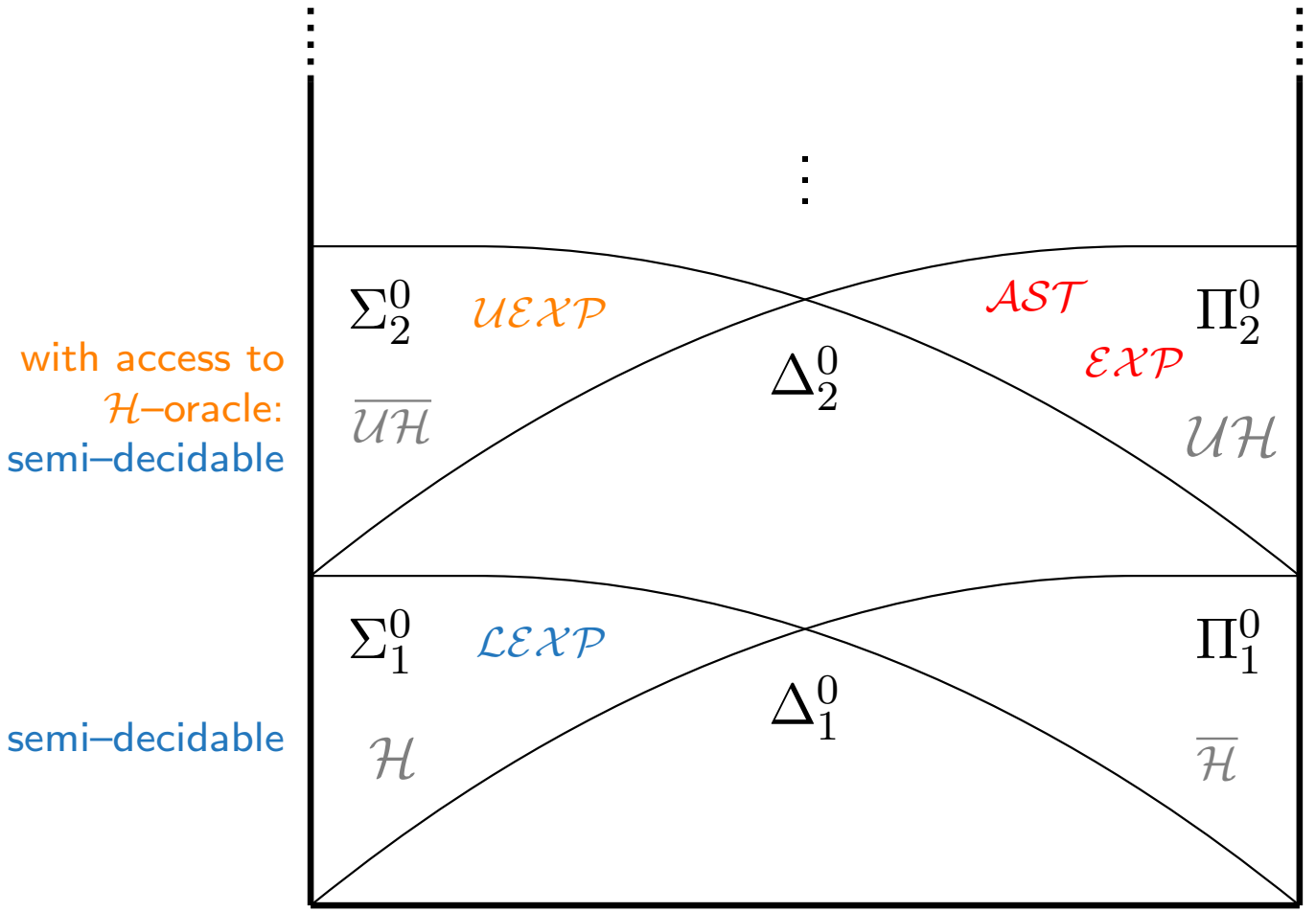
Summary


with access to
$\mathcal{H}$–oracle:
semi–decidable
semi–decidable
Σ_2^0
$\mathcal{UEXP}$
Π_2^0
Δ_2^0
$\overline{\mathcal{UH}}$
$\mathcal{UH}$
Σ_1^0
$\mathcal{LEXP}$
Π_1^0
Δ_1^0
$\mathcal{H}$
$\overline{\mathcal{H}}$

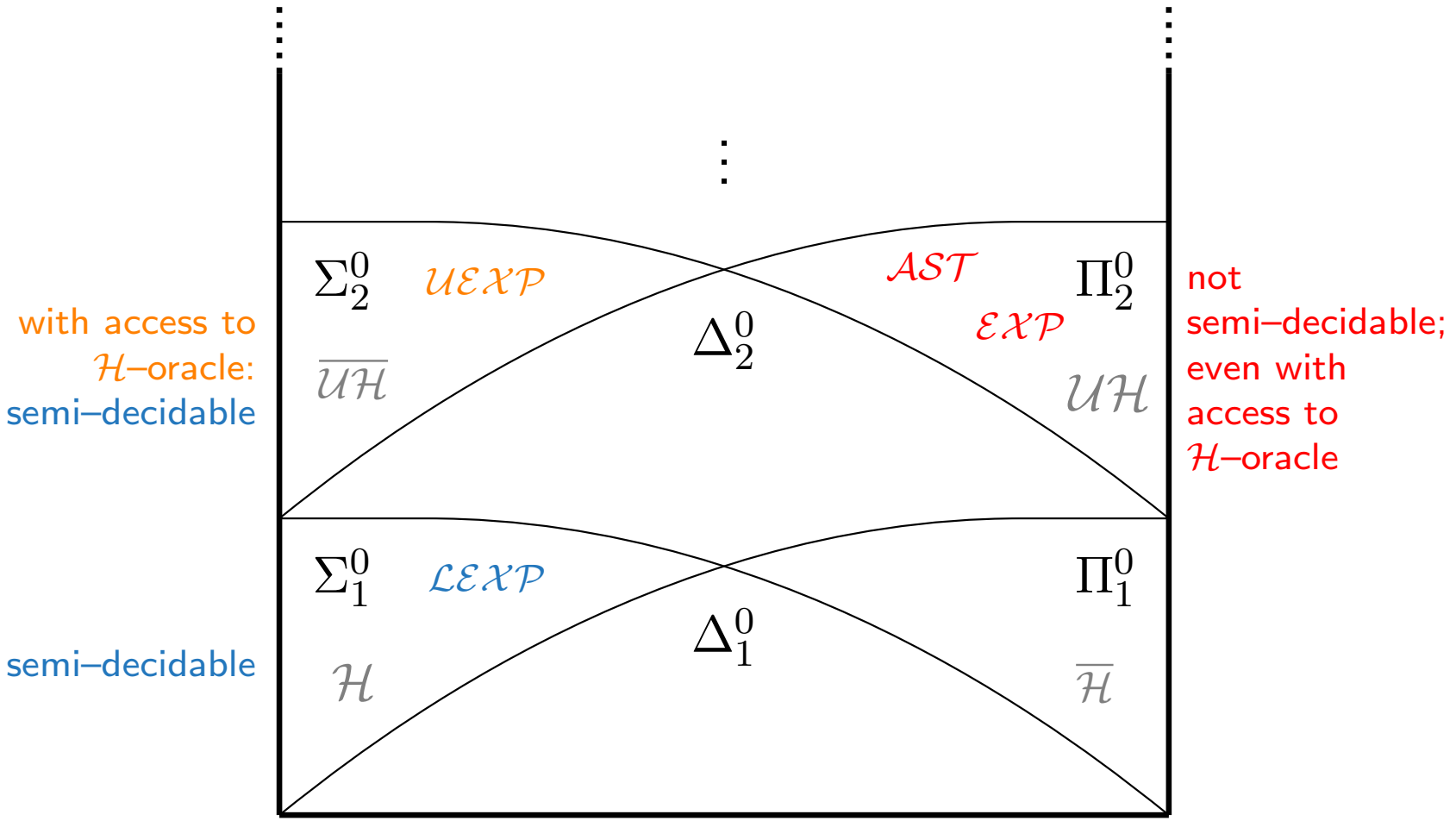
Summary


with access to
$\mathcal{H}$–oracle:
semi–decidable
semi–decidable
Σ_2^0
$\mathcal{UEXP}$
Δ_2^0
Π_2^0
$\mathcal{EXP}$
$\overline{\mathcal{UH}}$
$\mathcal{UH}$
Σ_1^0
$\mathcal{LEXP}$
Δ_1^0
Π_1^0
$\mathcal{H}$
$\overline{\mathcal{H}}$

Summary


with access to
$\mathcal{H}$–oracle:
semi–decidable
semi–decidable
Σ_2^0
$\mathcal{UEXP}$
$\mathcal{AST}$
Π_2^0
$\mathcal{EXP}$
Δ_2^0
$\overline{\mathcal{UH}}$
$\mathcal{UH}$
Σ_1^0
$\mathcal{LEXP}$
Π_1^0
Δ_1^0
$\mathcal{H}$
$\overline{\mathcal{H}}$

Summary


with access to $\mathcal{H}$–oracle: semi–decidable
semi–decidable
not semi–decidable; even with access to $\mathcal{H}$–oracle
Σ_2^0
$\mathcal{UEXP}$
$\mathcal{AST}$
Π_2^0
$\mathcal{EXP}$
Δ_2^0
$\overline{\mathcal{UH}}$
$\mathcal{UH}$
Σ_1^0
$\mathcal{LEXP}$
Π_1^0
Δ_1^0
$\mathcal{H}$
$\overline{\mathcal{H}}$

Summary

Thank you for your kind attention :-)