Comparing Different Projection Operators in the Cylindrical Algebraic Decomposition for SMT Solving

Tarik Viehmann, **Gereon Kremer**, Erika Ábrahám

SC² Workshop | Juli 29th
Outline

1 Preliminaries

2 CAD

3 Experiments
   Projections
   SMT solving
   Incompleteness of McCallum / Brown
   Effects of squarefree basis

4 Conclusion
Definition (Nonlinear arithmetic)

Boolean combinations of polynomial constraints over reals

Example:
\[ \exists x, y. x^2 + y^2 - 4 \leq 0 \land \left( x^2 - y + 0 < 0 \lor x^2 + 5 \cdot y + 5 < 0 \right) \]

Also:
- With quantifiers (NRA)
- Over integers (QF_NIA)
Definition (Nonlinear arithmetic)

Boolean combinations of polynomial constraints over reals

Example

\[ \exists x, y. \quad x^2 + y^2 - 4 \leq 0 \land (x^2 - y + 0.5 < 0 \lor x^2 + 5 \cdot y + 5 < 0) \]
Definition (Nonlinear arithmetic)

Boolean combinations of polynomial constraints over reals

Example

$$\exists x, y. \quad x^2 + y^2 - 4 \leq 0 \land (x^2 - y + 0.5 < 0 \lor x^2 + 5 \cdot y + 5 < 0)$$

Also:

- With quantifiers (NRA)
- Over integers (QF_NIA)
SMT Solving

Boolean abstraction
Tseitin’s transformation

quantifier-free FO formula

propositional logic formula in CNF

SAT solver

SAT or UNSAT

boolean assignment

theory constraints

SAT + model

or

UNSAT + reason

Theory solver
\[ \exists x, y. \quad x^2 + y^2 - 4 \leq 0 \land (x^2 - y + 0.5 < 0 \lor x^2 + 5 \cdot y + 5 < 0) \]
∃x, y. \( x^2 + y^2 - 4 \leq 0 \land (x^2 - y + 0.5 < 0 \lor x^2 + 5 \cdot y + 5 < 0) \)

Where are solutions?
\[ \exists x, y. \quad x^2 + y^2 - 4 \leq 0 \land (x^2 - y + 0.5 < 0 \lor x^2 + 5 \cdot y + 5 < 0) \]

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  - First dimension \( x \)

Cylindrical Algebraic Decomposition
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Where are solutions?
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- Where are solutions?
- What would a human do?
- What would CAD do?
  - First dimension \( x \)
  - Second dimension \( y \)
  - Test sample points
Cylindrical Algebraic Decomposition

\[ P_n \subseteq \mathbb{Z}[x_1, \ldots, x_n] \]

project

\[ P_{n-1} \subseteq \mathbb{Z}[x_1, \ldots, x_{n-1}] \]

project

\[ \vdots \]

project

\[ P_1 \subseteq \mathbb{Z}[x_1] \]
Cylindrical Algebraic Decomposition

\[ P_n \subseteq \mathbb{Z}[x_1, \ldots, x_n] \]

\[ P_{n-1} \subseteq \mathbb{Z}[x_1, \ldots, x_{n-1}] \]

\[ P_1 \subseteq \mathbb{Z}[x_1] \]

\[ Z_2 \subseteq \mathbb{Z}_1 \times \mathbb{R} \]

\[ Z_1 \subseteq \mathbb{R} \]

\[ Z_n \subseteq \mathbb{Z}_{n-1} \times \mathbb{R} \]

roots(\( P_n \) at \( Z_{n-1} \))

roots(\( P_3 \) at \( Z_2 \))

roots(\( P_2 \) at \( Z_1 \))

roots(\( P_1 \))

Comparing Different Projection Operators in the CAD for SMT Solving

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Cylindrical Algebraic Decomposition

\[ P_n \subseteq \mathbb{Z}[x_1, \ldots, x_n] \]

projection \[ P_{n-1} \subseteq \mathbb{Z}[x_1, \ldots, x_{n-1}] \]

projection \[ \vdots \]

projection \[ P_1 \subseteq \mathbb{Z}[x_1] \]

\[ Z_n \subseteq \mathbb{Z}_{n-1} \times \mathbb{R} \]

\[ \text{roots}(P_n \text{ at } Z_{n-1}) \]

\[ \vdots \]

\[ \text{roots}(P_3 \text{ at } Z_2) \]

\[ Z_2 \subseteq Z_1 \times \mathbb{R} \]

\[ \text{roots}(P_2 \text{ at } Z_1) \]

\[ Z_1 \subseteq \mathbb{R} \]

\[ \text{roots}(P_1) \]
**Intuition**

Cylinders in $\mathbb{R}^n$ based on the roots of $P_{n-1}$ form proper stacks. Substitute a sample from $\mathbb{R}^{n-1}$ into $P_n$, the roots cover all cylinders.
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We consider:

- Collins
- Hong
- McCallum
- Brown
Intuition

Cylinders in $\mathbb{R}^n$ based on the roots of $P_{n-1}$ form proper stacks. Substitute a sample from $\mathbb{R}^{n-1}$ into $P_n$, the roots cover all cylinders.

We consider:
- Collins
- Hong
- McCallum
- Brown

Not considered or specific use case:
- Lazard (improvement of McCallum)
- Seidl & Sturm (based on Hong for partial CAD)
- Strzeboński („local projection“)
- Brown & Košta („OneCell CAD“)
- ...

Comparing Different Projection Operators in the CAD for SMT Solving
Notation

Definition (Polynomials)

\[ p = \sum_{i=0}^{m} a_i \cdot x_i \text{ in main variable } x_n \text{ and } a_i \in \mathbb{R}[x_1, \ldots, x_{n-1}] \]

Definition (Simple properties)

\[ \text{coeffs}(p) := \{a_0, \ldots, a_m\} \quad \text{lcf}(p) := a_m \]

\[ \text{red}_k(p) := \sum_{i=0}^{m-k} a_i \cdot x_i \quad \text{red}(p) := \{\text{red}_k(p) \mid k = 0 \ldots m\} \]
Building blocks

\[ Syl(p, q) := \begin{vmatrix} a_k & \cdots & a_0 \\ a_k & \cdots & a_0 \\ \vdots & \ddots & \vdots \\ b_l & \cdots & b_0 \\ b_l & \cdots & b_0 \\ \vdots & \ddots & \vdots \\ b_l & \cdots & b_0 \end{vmatrix} \]

Definition (Principal subresultant coefficients)

\[ PCS(p, q) := \{ pcs_i \mid i = 0, \ldots, \min(k, l) \} \]
Building blocks

\[ M_j(p, q) := \begin{pmatrix}
  a_k & \cdots & a_0 \\
  a_k & \cdots & a_0 \\
  \vdots & \ddots & \vdots \\
  b_l & \cdots & b_0 \\
  b_l & \cdots & b_0 \\
  \vdots & \ddots & \vdots \\
  b_l & \cdots & b_0
\end{pmatrix} \]

\[ \left\{ \begin{array}{c}
l - j \\
k - j
\end{array} \right. \]
Building blocks

$M_j(p, q) := \begin{pmatrix} a_k & \cdots & a_0 \\ a_k & \cdots & a_0 \\ \vdots & \ddots & \vdots \\ b_l & \cdots & b_0 \\ b_l & \cdots & b_0 \\ \vdots & \ddots & \vdots \\ b_l & \cdots & b_0 \end{pmatrix}$

Definition (Principal subresultant coefficients)

$pcs_i(p, q) := \det(M_i)$

$PCS(p, q) := \{pcs_i \mid i = 0 \ldots \min(k, l)\}$
**Definition (Resultant)**

\[
res(p, q) := \det(Syl(p, q))
\]

\(p, q\) have a **common root** ⇔ \(res(p, q)\) has a root

**Definition (Discriminant)**

\[
disc(p) := res(p, p')
\]

\(p\) has a **multiple root** ⇔ \(disc(p)\) has a root
Definition (Collins' operator / Hong's operator)

\[
\begin{align*}
proj_1^C & := \bigcup_{p \in P} \bigcup_{r \in \text{red}(p)} \{ldcf(r)\} \cup \text{PSC}(r, r') \\
proj_2^C & := \bigcup_{p,q \in P} \bigcup_{r_p \in \text{red}(p)} \bigcup_{r_q \in \text{red}(q)} \text{PSC}(r_p, r_q) \\
proj_C & := \text{proj}_1^C \cup \text{proj}_2^C
\end{align*}
\]
Definition (Collins’ operator / Hong’s operator)

\[
\text{proj}_C^1 := \bigcup_{p \in P} \bigcup_{r \in \text{red}(p)} \{ldcf(r)\} \cup PSC(r, r')
\]

\[
\text{proj}_H^2 := \bigcup_{p, q \in P} \bigcup_{r_p \in \text{red}(p)} PSC(r_p, q)
\]

\[
\text{proj}_H := \text{proj}_C^1 \cup \text{proj}_H^2
\]
Definition (McCallum’s operator / Brown’s operator)

Let $P$ be a squarefree basis.

\[
\begin{align*}
\text{proj}_1^M & := \bigcup_{p \in P} \{\text{disc}(p)\} \cup \text{coefs}(p) \\
\text{proj}_2^M & := \bigcup_{p,q \in P} \{\text{res}(p, q)\} \\
\text{proj}^M & := \text{proj}_1^M \cup \text{proj}_2^M
\end{align*}
\]
Definition (McCallum’s operator / Brown’s operator)

Let \( P \) be a squarefree basis.

\[
\begin{align*}
\text{proj}_B^1 &:= \bigcup_{p \in P} \{\text{disc}(p)\} \cup \{\text{lcf}(p)\} \\
\text{proj}_M^2 &:= \bigcup_{p,q \in P} \{\text{res}(p,q)\} \\
\text{proj}_B &:= \text{proj}_B^1 \cup \text{proj}_M^2
\end{align*}
\]

Incomplete!
Experiments

- **SMT-RAT**
  - Projections: SAT + CAD
  - Solving: SAT + VS + CAD
  - No squarefree basis, no delineating polynomials (McCallum), no additional points (Brown)
  - But: fully incremental, early abort
Experiments

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- **QF_NRA from SMT-COMP 2014**

- **Timeout 60s**
Experiments

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- **Analyzed:**
  - Different projection operators
  - Different projection orders
Experiments

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  - Projections: SAT + CAD
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- **QF_NRA from SMT-COMP 2014**

- **Timeout 60s**

- **Analyzed:**
  - Different projection operators
  - Different projection orders

- **Not analyzed:**
  - Different variable orderings
  - Different lifting orders
Projection sizes

- Project **all polynomials**, ignore boolean structure
- 5698 benchmarks where all projections terminated
Projection sizes

- Project all polynomials, ignore boolean structure
- 5698 benchmarks where all projections terminated
- On average 6.4 polynomials of degree 5.2 (total degree 6.1)
- Rarely more than 5 variables
Project all polynomials, ignore boolean structure
5698 benchmarks where all projections terminated
On average 6.4 polynomials of degree 5.2 (total degree 6.1)
Rarely more than 5 variables

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Projection sizes

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▶ Theory: $\text{proj}_B \subseteq \text{proj}_M \subseteq \text{proj}_H \subseteq \text{proj}_C$
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- **Theory:** \( \text{proj}_B \subseteq \text{proj}_M \subseteq \text{proj}_H \subseteq \text{proj}_C \)

- **Hong** improves a lot upon Collins

- **McCallum** improves a lot upon Hong
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- Theory: \( \text{proj}_B \subseteq \text{proj}_M \subseteq \text{proj}_H \subseteq \text{proj}_C \)
- **Hong** improves a lot upon Collins
- **McCallum** improves a lot upon Hong
- **Brown** improves a bit, but more speedups in lifting phase
- Hong may be viable if **incompleteness** of McCallum is an issue
SMT solving performance

- 5698 benchmarks from before
- Incremental calls from SAT module
- Incremental projection, early abort if satisfying solution is found
5698 benchmarks from before

Incremental calls from SAT module

**Incremental projection**, early abort if **satisfying solution** is found

⇒ Size of projection may not be that crucial
SMT solving performance

- 5698 benchmarks from before
- Incremental calls from SAT module
- **Incremental projection**, early abort if **satisfying solution** is found
- ⇒ Size of projection may not be that crucial

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<td>399</td>
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McCallum vs. Brown

- Similar behaviour, but some outliers in both directions
Similar behaviour, but some outliers in both directions
Make behaviour different (run in parallel)
McCallum vs. Brown

- Similar behaviour, but some outliers in **both directions**
- Make behaviour different (run in parallel)
- Modify Brown: Consider **resultants last** for projection
Similar behaviour, but some outliers in both directions
Make behaviour different (→ run in parallel)
Modify Brown: Consider resultants last for projection
Incompleteness of McCallum / Brown

- McCallum and Brown are *incomplete*
- Is this a problem in *practice*?
Incompleteness of McCallum / Brown

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- 510 out of 5889 benchmarks
  (may be fixed by delineating polynomials or additional points)
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- 157 were found to be unsatisfiable
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- All are correct!
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- McCallum and Brown are **incomplete**
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- **510 out of 5889 benchmarks**
  (may be fixed by delineating polynomials or additional points)
- **353** were found to be satisfiable
- **157** were found to be unsatisfiable

- **All are correct!**

- ⇒ not a pressing issue **on our SMT benchmarks**
Effects of squarefree basis

- McCallum / Brown require $P_k$ to be a squarefree basis
- Difficult to compute ignored until now
Effects of squarefree basis

- McCallum / Brown require $P_k$ to be a **squarefree basis**
- **Difficult** to compute ignored until now

- Using CoCoALib
Effects of squarefree basis

- McCallum / Brown require $P_k$ to be a **squarefree basis**
- **Difficult** to compute ignored until now

- Using CoCoALib
- Overall solving is about **10% slower**
- **Less timeouts!** McCallum: 889 → 739, Brown: 842 → 739
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⇒ usually detrimental, sometimes essential
required for correctness!
Conclusion

- Overall trend matches theoretical expectation
- Individual examples **may vary wildly**
- ⇒ Portfolio?
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- Adapt variable ordering?
- Effects of delineating polynomials and additional points?