From Measure Theory to CTMDPs

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Measure Theory

Our Setting
Assume a set $\Omega$, called sample space. Subsets $A$ of $\Omega$ are called events.

Idea: Measure the \{size | probability | volume | length\} of events!

Intuition: Let $\omega \in \Omega$ be the outcome of an experiment. Then $A$ is an event if $\omega \in A$ can be decided.
Fields and $\sigma$–fields

Definition (Field)
A class of subsets $\mathcal{F}$ of $\Omega$ is a field iff

1. $\Omega \in \mathcal{F}$.
2. $A \in \mathcal{F} \implies A^c \in \mathcal{F}$.
3. $A_1, \ldots, A_n \in \mathcal{F} \implies \bigcup_{i=1}^{n} A_i \in \mathcal{F}$

where $n \in \mathbb{N}$
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Definition ($\sigma$–Field)
$\mathcal{F}$ is a $\sigma$–field iff it is closed under countable union:

$$A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$$
Fields and \( \sigma \)-fields

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   where \( n \in \mathbb{N} \).

Definition (\( \sigma \)-Field)
\( \mathcal{F} \) is a \( \sigma \)-field iff it is closed under countable union:

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\]

Let \( \mathcal{C} \subseteq 2^\Omega \). \( \sigma(\mathcal{C}) \) denotes the smallest \( \sigma \)-field containing \( \mathcal{C} \).
Example: The Borel $\sigma$–field

Define **right–semiclosed intervals** to be

- $(a, b]$ where $-\infty \leq a < b < +\infty$ and
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Finite Disjoint Unions
Define the class of finite disjoint unions of right–semiclosed intervals:

$$\mathcal{F}_0(\mathbb{R}) := \{I_1 \uplus I_2 \uplus \cdots \uplus I_n \mid n \in \mathbb{N}\}.$$
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Verify the properties of a field:

1. $\mathbb{R} = (-\infty, +\infty) \in \mathcal{F}_0(\mathbb{R})$. 
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3. If $I_1 \uplus \cdots \uplus I_n \in \mathcal{F}_0(\mathbb{R})$ and $J_1 \uplus \cdots \uplus J_n \in \mathcal{F}_0(\mathbb{R})$ then $(I_1 \uplus \cdots \uplus I_n) \cup (J_1 \uplus \cdots \uplus J_n) \in \mathcal{F}_0(\mathbb{R})$. 

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$\mathcal{F}_0(\mathbb{R})$ is a field.
Example: The Borel $\sigma$–field

**Borel $\sigma$–field**

Let $\mathcal{E} = \{(a, b] \mid a, b \in \mathbb{R} \text{ and } a < b\}$ and let $\sigma(\mathcal{E})$ denote the smallest $\sigma$–field containing $\mathcal{E}$. Then $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{E})$ is the **Borel $\sigma$–field**.
Example: The Borel \( \sigma \)-field

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Example

\( \mathcal{B}(\mathbb{R}) \) has many generators:

- \( \mathcal{F}_0(\mathbb{R}) \), the set of finite disjoint unions of right–semiclosed intervals,
- \( \mathcal{E}' = \{[a, b] \mid a, b \in \mathbb{R} \text{ and } a < b\} \),
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Intuition: Construct $\sigma$–field by forming countable unions and complements of intervals in all possible ways.
Measures

Intuition
Measure the “size” of sets in $\sigma$–field $\mathcal{F}$.
Notions of length, volume or probability.
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Notions of length, volume or probability.

Definition (Measure)
Let $\mathcal{F}$ be a $\sigma$–field over subsets of $\Omega$. A measure is a function

$$\mu : \mathcal{F} \rightarrow \bar{\mathbb{R}}_{\geq 0}$$

where $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$

which is countably additive:

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

for disjoint sets $A_i \in \mathcal{F}$. 
Introduction to Measure Theory

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Remark: If \( \mu(\Omega) = 1 \), \( \mu \) is a probability measure.
Example: A Measure on $\mathcal{B}(\mathbb{R})$

The size of intervals
Given interval $(a, b]$, $a < b \in \mathbb{R}$. Define its “length” as follows:

$$\mu(a, b] = b - a$$
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Sizes on the field $\mathcal{F}_0(\mathbb{R})$
On the set of finite disjoint unions of right–semiclosed intervals:
Let $I_1 \uplus \cdots \uplus I_n \in \mathcal{F}_0(\mathbb{R})$. Extend $\mu$ to $\mathcal{F}_0(\mathbb{R})$ by defining

$$\mu(I_1 \uplus \cdots \uplus I_n) = \sum_{i=1}^{n} \mu(I_i)$$
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**But:** What about $\mu(A)$ for **arbitrary** $A \in \mathcal{B}(\mathbb{R})$?
Extension of Measures

Motivation
Define **countably additive** set function $\mu$ on a field $\mathcal{F}_0$. Then extend it to the $\sigma$–field **by magic**.
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Define countably additive set function $\mu$ on a field $\mathcal{F}_0$. Then extend it to the $\sigma$–field by magic.

Theorem (Carathéodory Extension Theorem)
Let $\mathcal{F}_0$ be a field over subsets of a set $\Omega$ and let $\mu$ be a measure on $\mathcal{F}_0$. If $\mu$ is $\sigma$–finite, i.e.

$$\Omega = \bigcup_{i=1}^{\infty} A_i \quad \text{where } A_i \in \mathcal{F}_0 \text{ and } \mu(A_i) < \infty,$$

then $\mu$ has a unique extension to $\sigma(\mathcal{F}_0)$.
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In practice: Avoid the $\sigma$–field whenever possible!
There’s Still a Catch in it: Countable Additivity!

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Up to now, we defined the “length” $\mu$ on subclasses of $\mathcal{B}(\mathbb{R})$: 
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1. \( \mu(a, b] = b - a \) for right–semiclosed intervals
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1. $\mu(a, b] = b - a$ for right–semiclosed intervals
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3. **But**: For the extension from $\mathcal{F}_0(\mathbb{R})$ to $\mathcal{B}(\mathbb{R})$ by Carathéodory:

$$\mu(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j)$$

where $A_1, A_2, \cdots \in \mathcal{F}_0(\mathbb{R})$, $\bigcup_{j=1}^{\infty} A_j \in \mathcal{F}_0$ and the $A_j$ disjoint.
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Theorem

Let $F : \mathbb{R} \to \mathbb{R}$ be a distrib. function. Let $\mu(a, b] := F(b) - F(a)$. There is a unique extension of $\mu$ to a Lebesgue–Stieltjes measure on $\mathbb{R}$.

Thus: Countable additivity of $\mu$ follows by defining $F(x) := x$. 
Lebesgue’s Intuition

Lebesgue about his integral

“One might say that Riemann’s approach is comparable to a messy merchant who counts coins in the order they come to his hand whereas we act like a prudent merchant who says:

- I have $A_1$ coins à one crown, that is $A_1 \cdot 1$ crowns,
- $A_2$ coins à two crowns, that is $A_2 \cdot 2$ crowns and
- $A_3$ coins à five crowns, that is $A_3 \cdot 5$ crowns.

Therefore I have $A_1 \cdot 1 + A_2 \cdot 2 + A_3 \cdot 5$ crowns.

Both approaches – no matter how rich the merchant might be – lead to the same result since he only has to count a finite number of coins. But for us who must add infinitely many indivisibles, the difference between the approaches is essential.”

H. Lebesgue, 1926
Measurability

Definition (Measurability)

Let $\Omega_1, \Omega_2$ be sets with associated $\sigma$–fields $\mathcal{F}_1$ and $\mathcal{F}_2$. 
$h : \Omega_1 \rightarrow \Omega_2$ is measurable iff 

$$h^{-1}(A) \in \mathcal{F}_1 \quad \text{for each } A \in \mathcal{F}_2$$

Notation: $h : (\Omega_1, \mathcal{F}_1) \rightarrow (\Omega_2, \mathcal{F}_2)$. 
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Some remarks:

- $h$ is Borel measurable if $h : (\Omega, \mathcal{F}) \to (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$. 
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Notation: $h : (\Omega_1, \mathcal{F}_1) \to (\Omega_2, \mathcal{F}_2)$.

Some remarks:
- $h$ is Borel measurable if $h : (\Omega, \mathcal{F}) \to (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}}))$.
- In probability theory, $h$ is called a random variable.
Simple Functions

Definition (Simple Functions)
Let $h : \Omega \to \bar{\mathbb{R}}$. $h$ is simple iff

1. $h$ is measurable and
2. takes on only finitely many values.
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Let \( h : \Omega \to \bar{\mathbb{R}} \). \( h \) is **simple** iff

1. \( h \) is measurable and
2. takes on only finitely many values.

If \( h \) is a **simple** function, it can be represented as

\[
h(\omega) := \sum_{i=1}^{n} x_i \cdot I_{A_i}(\omega)
\]

where \( A_i \in \mathcal{F} \) are pairwisely disjoint.

\( I_{A_i} \) denotes the indicator function \( I_{A_i}(\omega) := \begin{cases} 1 & \text{if } \omega \in A_i \\ 0 & \text{otherwise} \end{cases} \).
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$I_{A_i}$ denotes the indicator function $I_{A_i}(\omega) := \begin{cases} 1 & \text{if } \omega \in A_i \\ 0 & \text{otherwise} \end{cases}$.

**Intuition:** Choose $A_i$ as the *preimage* of $x_i$!
Lebesgue Integral

Definition (Lebesgue Integral for Simple Functions)
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, $h : \Omega \to \mathbb{R}$ simple:

$$h(\omega) := \sum_{i=1}^{n} x_i \cdot I_{A_i}(\omega)$$

where the $A_i$ are disjoint sets in $\mathcal{F}$. 
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Let \((\Omega, \mathcal{F}, \mu)\) be a measure space, \(h : \Omega \to \overline{\mathbb{R}}\) simple:

\[ h(\omega) := \sum_{i=1}^{n} x_i \cdot I_{A_i}(\omega) \]
where the \(A_i\) are disjoint sets in \(\mathcal{F}\).

The Lebesgue–integral of \(h\) is defined as

\[ \int_{\Omega} h \, d\mu := \sum_{i=1}^{n} x_i \cdot \mu(A_i). \]

Intuition: Multiply each \(x_i\) with the measure of its preimage \(A_i\).
Example: Lebesgue Integral

\[ \mu(A_1) = \mu(\text{green}) \]
\[ \mu(A_3) = \mu(\text{yellow}) \]
\[ \mu(A_2) = \mu(\text{red}) \]
\[ \mu(A_4) = \mu(\text{purple}) \]
Example: Lebesgue Integral

\[ \int_{\Omega} h \, d\mu = x_1 \mu(A_1) + x_2 \mu(A_2) + x_3 \mu(A_3) \]
Example: Riemann (Darboux) Integral
Lebesgue Integral on Nonnegative Functions

**Definition**
If $h$ is **nonnegative Borel measurable**, then

$$\int_{\Omega} h \, d\mu := \sup \left\{ \int_{\Omega} s \, d\mu \mid s \text{ is simple and } 0 \leq s \leq h \right\}.$$ 

**Theorem**
A **nonnegative Borel measurable function** $h$ **is the limit of an increasing sequence of nonnegative simple functions** $h_n$. 
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Finite Product Spaces

Definition (Product Space)
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Definition (Product Space)
Let $(\Omega_j, \mathcal{F}_j)$ be a measurable space, $j = 1, \ldots, n$. Then

- $\Omega = \Omega_1 \times \cdots \times \Omega_n$
- $A = A_1 \times A_2 \times \cdots \times A_n$ is a **measurable rectangle** if $A_j \in \mathcal{F}_j$. 


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- \(A = A_1 \times A_2 \times \cdots \times A_n\) is a **measurable rectangle** if \(A_j \in \mathcal{F}_j\).
- The set of measurable rectangles is denoted \(\mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_n\).
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- The set of measurable rectangles is denoted

\[\mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_n.\]

- The **product \(\sigma\)-field** \(\mathcal{F}\) is the smallest \(\sigma\)-field containing all measurable rectangles:

\[\mathcal{F} := \sigma\left(\mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_n\right)\]
To start with: Only products of two $\sigma$–fields!

Preparation

Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ be a measure space, $\mu_1$ $\sigma$–finite on $\mathcal{F}_1$. 
Measures on Finite Product Spaces

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Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ be a measure space, $\mu_1$ $\sigma$–finite on $\mathcal{F}_1$. Further let $\mathcal{F}_2$ be a $\sigma$–field over subsets of $\Omega_2$. 
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$$\mu(\omega_1, \cdot) : \mathcal{F}_2 \rightarrow \mathbb{R}$$

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2. Borel measurable in $\omega_1$ and
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Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ be a measure space, $\mu_1$ $\sigma$–finite on $\mathcal{F}_1$. Further let $\mathcal{F}_2$ be a $\sigma$–field over subsets of $\Omega_2$. Assume that for each $\omega_1 \in \Omega_1$ we have a function

$$\mu(\omega_1, \cdot) : \mathcal{F}_2 \to \bar{\mathbb{R}}$$

which is

1. a measure on $\mathcal{F}_2$,
2. Borel measurable in $\omega_1$ and
3. uniformly $\sigma$–finite:

$$\Omega_2 = \bigcup_{n=1}^{\infty} B_n$$

where $\mu(\omega_1, B_n) \leq k_n$ for all $\omega_1$ and fixed $k_n \in \mathbb{R}$. 

Martin Neuhäußer (MOVES)  From Measure Theory to CTMDPs  November 9th, 2006  22 / 1
Measures on Finite Product Spaces

**Theorem (Product Measure Theorem)**

*Given $(\Omega_1, \mathcal{F}_1, \mu_1), (\Omega_2, \mathcal{F}_2)$ and $\mu(\omega_1, \cdot)$ as before.*
Theorem (Product Measure Theorem)

Given $(\Omega_1, \mathcal{F}_1, \mu_1)$, $(\Omega_2, \mathcal{F}_2)$ and $\mu(\omega_1, \cdot)$ as before. There is a unique measure $\mu$ on $\mathcal{F}$ such that on $\mathcal{F}_1 \times \mathcal{F}_2$:

$$\mu(A \times B) = \int_A \mu(\omega_1, B) \mu_1(d\omega_1).$$
Theorem (Product Measure Theorem)

Given \((\Omega_1, \mathcal{F}_1, \mu_1), (\Omega_2, \mathcal{F}_2)\) and \(\mu(\omega_1, \cdot)\) as before. There is a unique measure \(\mu\) on \(\mathcal{F}\) such that on \(\mathcal{F}_1 \times \mathcal{F}_2\):

\[
\mu(A \times B) = \int_A \mu(\omega_1, B) \, \mu_1(d\omega_1).
\]

\(\mu\) is defined (now on the entire \(\sigma\)-field) as follows:

\[
\mu(F) := \int_{\Omega_1} \mu(\omega_1, F(\omega_1)) \, \mu_1(d\omega_1), \quad \text{for all } F \in \mathcal{F}
\]

where \(F(\omega_1) := \{\omega_2 \mid (\omega_1, \omega_2) \in F\}\).
Theorem (Fubini’s Theorem)

Let \( f : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})) \). If \( f \) is nonnegative, then

\[
\int_{\Omega_2} f(\omega_1, \omega_2) \, \mu(\omega_1, d\omega_2)
\]

exists and defines a **Borel measurable** function.
Theorem (Fubini’s Theorem)

Let \( f : (\Omega, \mathcal{F}) \rightarrow (\bar{\mathbb{R}}, \mathcal{B}(\bar{\mathbb{R}})) \). If \( f \) is nonnegative, then

\[
\int_{\Omega_2} f(\omega_1, \omega_2) \mu(\omega_1, d\omega_2)
\]

exists and defines a Borel measurable function. Also

\[
\int_{\Omega} f \, d\mu = \int_{\Omega_1} \left( \int_{\Omega_2} f(\omega_1, \omega_2) \mu(\omega_1, d\omega_2) \right) \mu_1(d\omega_1).
\]

Justification of iterated integration!
Extension to Larger Product Spaces

Now, consider products of more than two $\sigma$–fields!

Preparation

Let $\mathcal{F}_j$ be a $\sigma$–field of subsets of $\Omega_j$, $j = 1, \ldots, n$. 
Introduction to Measure Theory

Measures on Product Spaces

Extension to Larger Product Spaces

Now, consider products of more than two \( \sigma \)-fields!

Preparation

Let \( \mathcal{F}_j \) be a \( \sigma \)-field of subsets of \( \Omega_j, \ j = 1, \ldots, n \).
Let \( \mu_1 \) be a \( \sigma \)-finite measure on \( \mathcal{F}_1 \).
Now, consider products of more than two σ–fields!

Preparation

Let $\mathcal{F}_j$ be a σ–field of subsets of $\Omega_j$, $j = 1, \ldots, n$. Let $\mu_1$ be a σ–finite measure on $\mathcal{F}_1$ and assume that for each $(\omega_1, \ldots, \omega_j)$ we have a function

$$\mu(\omega_1, \omega_2, \ldots, \omega_j, \cdot) : \mathcal{F}_{j+1} \to \bar{\mathbb{R}}$$

which is
Now, consider products of more than two $\sigma$–fields!

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Now, consider products of more than two $\sigma$–fields!

**Preparation**

Let $\mathcal{F}_j$ be a $\sigma$–field of subsets of $\Omega_j$, $j = 1, \ldots, n$. Let $\mu_1$ be a $\sigma$–finite measure on $\mathcal{F}_1$ and assume that for each $(\omega_1, \ldots, \omega_j)$ we have a function

$$\mu(\omega_1, \omega_2, \ldots, \omega_j, \cdot) : \mathcal{F}_{j+1} \rightarrow [0, \infty]$$

which is

1. a measure on $\mathcal{F}_{j+1}$ and
2. is measurable, i.e. for all fixed $C \in \mathcal{F}_{j+1}$:

$$\mu(\omega_1, \ldots, \omega_j, C) : (\Omega_1 \times \cdots \times \Omega_j, \sigma(\mathcal{F}_1 \times \cdots \times \mathcal{F}_j)) \rightarrow ([0, \infty], \mathcal{B}([0, \infty]))$$

3. uniformly $\sigma$–finite.
Measures on Larger Product Spaces

Theorem (Product Measure Theorem)

*There is a unique measure \( \mu \) on \( \mathcal{F} \) such that on \( \mathcal{F}_1 \times \cdots \times \mathcal{F}_n \):

\[
\mu(A_1 \times \cdots \times A_n) = \int_{A_1} \mu_1(d\omega_1) \int_{A_2} \mu(\omega_1, d\omega_2) \cdots \int_{A_{n-1}} \mu(\omega_1, \ldots, \omega_{n-2}, d\omega_{n-1}) \int_{A_n} \mu(\omega_1, \ldots, \omega_{n-1}, d\omega_n).
\]
Measures on Larger Product Spaces

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$$

Let $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. If $f \geq 0$, then

$$
\int_{\Omega} f \ d\mu = \int_{\Omega_1} \mu_1(d\omega_1) \int_{\Omega_2} \mu(\omega_1, d\omega_2) \cdot \cdots \int_{\Omega_n} f(\omega_1, \ldots, \omega_n) \mu(\omega_1, \ldots, \omega_{n-1}, d\omega_n).
$$
Definition (Cylinder Set)

Let $(\Omega_j, \mathcal{F}_j)$ be a measurable space, $j = 1, 2, \ldots$.

Let $\Omega = \prod_{j=1}^{\infty} \Omega_j$. If $B^n \subseteq \Omega_1 \times \cdots \times \Omega_n$, define

$$B_n := \{\omega \in \Omega \mid (\omega_1, \omega_2, \ldots, \omega_n) \in B^n\}.$$
Measures on Infinite Product Spaces

Definition (Cylinder Set)

Let $(\Omega_j, \mathcal{F}_j)$ be a measurable space, $j = 1, 2, \ldots$. Let $\Omega = \times_{j=1}^{\infty} \Omega_j$. If $B^n \subseteq \Omega_1 \times \cdots \times \Omega_n$, define

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$B_n$ is called **cylinder** with **base** $B^n$. 
Measures on Infinite Product Spaces

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- \( B_n \) is **measurable** if \( B^n \in \sigma(\mathcal{F}_1 \times \cdots \times \mathcal{F}_n) \).
Measures on Infinite Product Spaces

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\(B_n\) is called **cylinder** with base \(B^n\).

- \(B_n\) is **measurable** if \(B^n \in \sigma(\mathcal{F}_1 \times \cdots \times \mathcal{F}_n)\).
- \(B_n\) is a **rectangle** if \(B^n = A_1 \times \cdots \times A_n\) and \(A_j \subseteq \Omega_j\);
- \(B_n\) is a **measurable rectangle** if \(A_j \in \mathcal{F}_j\).
Measures on Infinite Product Spaces

Ionescu–Tulcea Theorem
Let $P_1$ be a probability measure on $\mathcal{F}_1$ and for each $(\omega_1, \ldots, \omega_j)$, $j \in \mathbb{N}$, assume a measurable probability measure $P(\omega_1, \ldots, \omega_j, \cdot)$ on $\mathcal{F}_{j+1}$.

Let $P_n$ be defined on $\sigma(\mathcal{F}_1 \times \cdots \times \mathcal{F}_n)$:

$$P_n(F) = \int_{\Omega_1} P_1(d\omega_1) \int_{\Omega_2} P(\omega_1, d\omega_2) \cdots \int_{\Omega_n} I_F(\omega_1, \ldots, \omega_n) \, P(\omega_1, \ldots, \omega_{n-1}, d\omega_n).$$
Measures on Infinite Product Spaces

Ionescu–Tulcea Theorem
Let $P_1$ be a probability measure on $\mathcal{F}_1$ and for each $(\omega_1, \ldots, \omega_j), j \in \mathbb{N}$, assume a measurable probability measure $P(\omega_1, \ldots, \omega_j, \cdot)$ on $\mathcal{F}_{j+1}$.

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There is a unique prob. measure $P$ on $\sigma\left(\prod_{j=1}^{\infty} \mathcal{F}_j\right)$ such that for all $n$:

$$P \{ \omega \in \Omega \mid (\omega_1, \ldots, \omega_n) \in B^n \} = P_n(B^n)$$
Measures on Infinite Product Spaces

Ionescu–Tulcea Theorem

Let \( P_1 \) be a probability measure on \( \mathcal{F}_1 \) and for each \( (\omega_1, \ldots, \omega_j), j \in \mathbb{N}, \) assume a measurable probability measure \( P(\omega_1, \ldots, \omega_j, \cdot) \) on \( \mathcal{F}_{j+1}. \)

Let \( P_n \) be defined on \( \sigma(\mathcal{F}_1 \times \cdots \times \mathcal{F}_n) : \)

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P_n(F) = \int_{\Omega_1} P_1(d\omega_1) \int_{\Omega_2} P(\omega_1, d\omega_2) \cdots \int_{\Omega_n} I_F(\omega_1, \ldots, \omega_n) P(\omega_1, \ldots, \omega_{n-1}, d\omega_n).
\]

There is a unique prob. measure \( P \) on \( \sigma \left( \times_{j=1}^{\infty} \mathcal{F}_j \right) \) such that for all \( n : \)

\[
P \{ \omega \in \Omega \mid (\omega_1, \ldots, \omega_n) \in B^n \} = P_n(B^n)
\]

Intuition: The measure of a cylinder equals the measure of its finite base.
Continuous Time Markov Decision Processes

Definition

A CTMDP is a tuple $\mathcal{C} = (S, \text{Act}, R, \text{AP}, L)$ with finite set of states $S$, labeled according to $\text{AP}$ and $L$. Further

- $\text{Act}$ is the set of possible actions and
- $R : S \times \text{Act} \times S \rightarrow \mathbb{R}$ is a transition rate matrix, parameterized with actions.
Continuous Time Markov Decision Processes

**Definition**
A CTMDP is a tuple $\mathcal{C} = (S, \text{Act}, R, \text{AP}, L)$ with finite set of states $S$, labeled according to AP and L. Further

- $\text{Act}$ is the set of possible actions and
- $R : S \times \text{Act} \times S \to \mathbb{R}$ is a transition rate matrix, parameterized with actions.

**Example**
Being in state $s \in S$,

1. choose enabled action from $\text{Act}(s)$
2. sojourn time in $s$: $1 - e^{-E(s,a)t}$
3. next state probability: $\frac{R(s,a,s')}{E(s,a)}$

where $E(s,a) := \sum_{s' \in S} R(s,a,s')$. 
Paths in a CTMDP

Definition (Paths)
Let $\mathcal{C} = (\mathcal{S}, \mathcal{A}, \mathcal{R}, \mathcal{AP}, \mathcal{L})$. Finite paths are denoted

$$\pi = s_0 \xrightarrow{a_0,t_0} s_1 \xrightarrow{a_1,t_1} s_2 \xrightarrow{a_2,t_2} \cdots \xrightarrow{a_{n-1},t_{n-1}} s_n.$$
Paths in a CTMDP

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\pi = s_0 \xrightarrow{a_0,t_0} s_1 \xrightarrow{a_1,t_1} s_2 \xrightarrow{a_2,t_2} \cdots \xrightarrow{a_{n-1},t_{n-1}} s_n.
$$

Sets of paths are denoted as usual:

$$
\text{Paths}^n := \mathcal{S} \times (\text{Act} \times \mathbb{R} \times \mathcal{S})^n \quad \text{and} \quad \text{Paths}^* := \bigcup_{i=0}^{\infty} \text{Paths}^n
$$
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Some notation

- $|\pi| := n$ and $\pi \downarrow := s_n$
Paths in a CTMDP

Definition (Paths)

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Some notation

- $|\pi| := n$ and $\pi \downarrow := s_n$
- $\pi[i..j] := s_i \xrightarrow{a_i,t_i} \cdots \xrightarrow{a_{j-1},t_{j-1}} s_j$ for $0 \leq i < j \leq |\pi|$. 
Paths in a CTMDP

Definition (Paths)
Let \( \mathcal{C} = (\mathcal{S}, \text{Act}, \mathcal{R}, \text{AP}, \mathcal{L}) \). Finite paths are denoted

\[\pi = s_0 \xrightarrow{a_0,t_0} s_1 \xrightarrow{a_1,t_1} s_2 \xrightarrow{a_2,t_2} \cdots \xrightarrow{a_{n-1},t_{n-1}} s_n.\]

Sets of paths are denoted as usual:

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Some notation

- \(|\pi| := n \) and \( \pi \downarrow := s_n \)
- \( \pi[i..j] := s_i \xrightarrow{a_i,t_i} \cdots \xrightarrow{a_{j-1},t_{j-1}} s_j \) for \( 0 \leq i < j \leq |\pi| \).
- \( \pi @ t \) is the state occupied in \( \pi \) at time \( t \).
- \( \delta(\pi, n) = t_n \) denotes the time spent in the \( n \)-th state.
Paths in a CTMDP

Example

\[ \pi \otimes t := \delta \left( \pi, \min \{ k \in \mathbb{N} \mid \sum_{i=0}^{k} t_i > t \} \right) \]

\[ \delta(\pi, n) := t_n \]
Paths in a CTMDP

Example

\[ \pi @ t = s_0 \]

\[ \pi @ t := \delta \left( \pi, \min\{k \in \mathbb{N} \mid \sum_{i=0}^{k} t_i > t \} \right) \]

\[ \delta(\pi, n) := t_n \]
Paths in a CTMDP

Example

\[
\pi \circ t = s_0 \quad \pi \circ t = s_4
\]

\[
\pi \circ t := \delta \left( \pi, \min \{ k \in \mathbb{N} \mid \sum_{i=0}^{k} t_i > t \} \right) \quad \delta(\pi, n) := t_n
\]
Paths in a CTMDP

**Example**

\[ \pi \at t = s_0 \quad \delta(\pi, 2) = t_2 \quad \pi \at t = s_4 \]

\[ \pi \at t := \delta\left(\pi, \min\{k \in \mathbb{N} \mid \sum_{i=0}^{k} t_i > t\}\right) \quad \delta(\pi, n) := t_n \]
Paths in a CTMDP

Definition (Infinite Paths)
The set of infinite paths is

\[ \text{Paths}^\omega := S \times (\text{Act} \times \mathbb{R} \times S)^\omega . \]

The definitions are extended to \( \text{Paths}^\omega \) if appropriate.
In CTMDP, the next action is chosen \textit{nondeterministically}. \(\rightarrow\) Nondeterminism must be resolved to assign probabilities.

\textbf{Classes of schedulers}

A scheduler resolves the nondeterminism in a \textit{CTMDP}. According to the information available, distinguish:

1. information about the history:
   - stationary markovian, markovian deterministic, history dependent
2. timed or time–abstract

The decision taken can either be

1. deterministic
2. randomized
Example: Scheduler Classes

Assume $C = (S, \text{Act}, R, \text{AP}, L)$.

1. **Stationary Markovian deterministic scheduler:**
   Consider $SMD$–scheduler that always chooses action $b$:

   $$D : S \rightarrow \text{Act} : s \mapsto b$$

   $C$ and $D$ induce a CTMC as follows:
Example: Scheduler Classes

Assume \( \mathcal{C} = (S, \text{Act}, R, \text{AP}, L) \).

1. **Stationary Markovian deterministic scheduler:**
   Consider \( SMD \)–scheduler that always chooses action \( b \):
   
   \[
   D : S \rightarrow \text{Act} : s \mapsto b
   \]

   \( \mathcal{C} \) and \( D \) induce a CTMC as follows:

   \[
   \begin{array}{c}
   \rightarrow s_0 \\
   s_0 \\
   s_1 \\
   s_2 \\
   s_3 \\
   \end{array}
   \]

   \[
   \begin{array}{c}
   b, 0.5 \\
   b, 15 \\
   b, 5 \\
   a, 0.5 \\
   a, 0.1 \\
   \end{array}
   \]
Example: Scheduler Classes

Assume $C = (S, \text{Act}, R, \text{AP}, L)$.

1. **Stationary Markovian deterministic scheduler:**
   Consider $SMD$—scheduler that always chooses action $b$:
   \[ D : S \rightarrow \text{Act} : s \mapsto b \]

   $C$ and $D$ induce a CTMC as follows:

   \[ s_0 \xrightarrow{0.5} s_1 \]
Example: Scheduler Classes

Assume $\mathcal{C} = (S, \text{Act}, R, \text{AP}, L)$.

1. **Stationary Markovian deterministic scheduler:**

   Consider $SMD$–scheduler that always chooses action $b$:

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Assume $\mathcal{C} = (\mathcal{S}, \mathcal{Act}, \mathcal{R}, \mathcal{AP}, \mathcal{L})$.

1. **Stationary Markovian deterministic scheduler:**
   Consider $SMD$–scheduler that always chooses action $b$:
   \[ D : \mathcal{S} \rightarrow \mathcal{Act} : s \mapsto b \]

   $\mathcal{C}$ and $D$ induce a CTMC as follows:

   - $s_0$ to $s_1$: 0.5
   - $s_1$ to $s_2$: 15
   - $s_1$ to $s_3$: 5
   - $s_2$ to $s_3$: 0.5
   - $s_0$ to $s_1$: $b$, 0.5
   - $s_1$ to $s_0$: $b$, 15
   - $s_1$ to $s_2$: $b$, 0.5
   - $s_2$ to $s_3$: $a$, 0.5
   - $s_3$ to $s_0$: $a$, 0.1
Example: Markovian Randomized Scheduler

Assume $\mathcal{C} = (\mathcal{S}, \text{Act}, \mathbb{R}, \text{AP}, L)$.

2 Markovian randomized scheduler:

\[
\mathcal{D} : \mathbb{N} \times \mathcal{S} \rightarrow \text{Distr}(\text{Act}) : (n, s) \mapsto \begin{cases} 
I_b & \text{if } s \in \{s_0, s_1\} \\
\gamma_n & \text{if } s = s_2
\end{cases}
\]

\[
\gamma_n : \text{Act} \rightarrow [0, 1] : x \mapsto \begin{cases} 
1 - 2^{-n} & \text{if } x = a \\
2^{-n} & \text{if } x = b
\end{cases}
\]

where

Diagram:

- $s_0$ to $s_1$: $b, 0.5$
- $s_1$ to $s_2$: $b, 15$
- $s_1$ to $s_3$: $b, 0.5$
- $s_2$ to $s_3$: $a, 0.5$
- $s_2$ to $s_1$: $b, 5$
- $s_3$ to $s_0$: $a, 0.1$
Example: Markovian Randomized Scheduler

Assume $C = (S, \text{Act}, R, AP, L)$.

1. Markovian randomized scheduler:

   $$D : \mathbb{N} \times S \rightarrow \text{Distr} (\text{Act}) : (n, s) \mapsto \begin{cases} I_b & \text{if } s \in \{s_0, s_1\} \\ \gamma_n & \text{if } s = s_2 \end{cases}$$

   $$\gamma_n : \text{Act} \rightarrow [0, 1] : x \mapsto \begin{cases} 1 - 2^{-n} & \text{if } x = a \\ 2^{-n} & \text{if } x = b \end{cases}$$

   where

   $s_0, 0$
Assume $\mathcal{C} = (\mathcal{S}, \text{Act}, \mathbb{R}, \text{AP}, L)$.

2 Markovian randomized scheduler:

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$$\gamma_n : \text{Act} \rightarrow [0, 1] : x \mapsto \begin{cases} 1 - 2^{-n} & \text{if } x = a \\ 2^{-n} & \text{if } x = b \end{cases}$$

where

- $s_0 \rightarrow s_1$ with probability 0.5
- $s_1 \rightarrow s_2$ with probability 0.5, $s_1 \rightarrow s_3$ with probability 0.5
- $s_2 \rightarrow s_1$ with probability 0.5, $s_2 \rightarrow s_3$ with probability 15
- $s_3 \rightarrow s_1$ with probability 0.5, $s_3 \rightarrow s_2$ with probability 0.5, $s_3 \rightarrow s_3$ with probability 0.5
- $s_0$ is an absorbing state

---

Example: Markovian Randomized Scheduler

Martin Neuhäuser (MOVES)
Example: Markovian Randomized Scheduler

Assume $\mathcal{C} = (\mathcal{S}, \text{Act}, \mathbb{R}, \text{AP}, \mathbb{L})$.

2. Markovian randomized scheduler:

\[ \mathcal{D} : \mathbb{N} \times \mathcal{S} \rightarrow \text{Distr}(\text{Act}) : (n, s) \mapsto \begin{cases} \mathbf{I}_b & \text{if } s \in \{s_0, s_1\} \\ \gamma_n & \text{if } s = s_2 \end{cases} \]

\[ \gamma_n : \text{Act} \rightarrow [0, 1] : x \mapsto \begin{cases} 1 - 2^{-n} & \text{if } x = a \\ 2^{-n} & \text{if } x = b \end{cases} \]

where

\[ s_0 \]
\[ s_1 \]
\[ s_2 \]
\[ s_3 \]
Assume $\mathcal{C} = (\mathcal{S}, \text{Act}, \mathbb{R}, \text{AP}, \mathbb{L})$.

Markovian randomized scheduler:

$$\mathcal{D} : \mathbb{N} \times \mathcal{S} \to \text{Distr} (\text{Act}) : (n, s) \mapsto \begin{cases} I_b & \text{if } s \in \{s_0, s_1\} \\ \gamma_n & \text{if } s = s_2 \end{cases}$$

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where

- $s_0, s_1, s_2, s_3$

- $b, 0.5$

- $b, 0.5$

- $a, 0.1$

- $a, 0.5$

- $b, 15$

- $b, 5$

- $\frac{3}{4} \cdot 0.5$

- $\frac{1}{4} \cdot 0.5$

- $0.5$

- $15$

- $5$

- $b, 0.5$

- $a, 0.1$

- $b, 15$

- $b, 5$

- $\frac{3}{4} \cdot 0.5$

- $\frac{1}{4} \cdot 0.5$

- $0.5$

- $15$

- $5$
Example: Markovian Randomized Scheduler

Assume $\mathcal{C} = (\mathcal{S}, \text{Act}, \mathbb{R}, \text{AP}, \mathbb{L})$.

2 Markovian randomized scheduler:

\[
\mathcal{D} : \mathbb{N} \times \mathcal{S} \rightarrow \text{Distr}(\text{Act}) : (n, s) \mapsto \begin{cases} 
I_b & \text{if } s \in \{s_0, s_1\} \\
\gamma_n & \text{if } s = s_2
\end{cases}
\]

where

\[
\gamma_n : \text{Act} \rightarrow [0, 1] : x \mapsto \begin{cases} 
1 - 2^{-n} & \text{if } x = a \\
2^{-n} & \text{if } x = b
\end{cases}
\]
**Example: The General Setting**

In general: CTMDP $C$ and scheduler $D$ do **not** induce a finite (or countable) CTMC!

**Example**

Consider a timed–history dependent scheduler. The states of its induced CTMC consist of the set of timed paths which is uncountable.
Given $\pi \in \text{Paths}^*$, the probability to continue by $\pi \xrightarrow{\alpha, t} s$ depends on

- $R(\pi \downarrow, a, s)$, the exponential distribution of CTMDP and
- $D(\pi, a)$, the scheduler’s decision.
Semantics: Combined Transitions

Given $\pi \in Paths^*$, the probability to continue by $\pi \downarrow_{a,t} s$ depends on

- $R(\pi \downarrow, a, s)$, the exponential distribution of CTMDP and
- $D(\pi, a)$, the scheduler’s decision.

Definition (Combined Transition)

Let $\Omega = Act \times \mathbb{R} \times S$. Then $(a, t, s) \in \Omega$ is a combined transition.

- $\mathcal{F}_{Act} \times \mathcal{B}(\mathbb{R}) \times \mathcal{F}_S$ is the class of measurable rectangles and
- $\mathcal{F} := \sigma\left(\mathcal{F}_{Act} \times \mathcal{B}(\mathbb{R}) \times \mathcal{F}_S\right)$ is the $\sigma$–field over combined transitions.
Semantics: Combined Transitions

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Example

$\mathcal{F}$ is the class of measurable sets of combined transitions;
$M \in \mathcal{F}$ is a set of combined transitions.
Semantics: From Paths to Rectangles

Finite Measurable Path Rectangles

A set of path of length $n$, represented as a Cartesian product

$$S_0 \times A_0 \times I_0 \times S_1 \times \cdots \times A_{n-1} \times I_{n-1} \times S_n$$
Semantics: From Paths to Rectangles

Finite Measurable Path Rectangles

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$$
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is called

- **path rectangle** iff $S_0 \subseteq S$ and $M_i \subseteq \Omega$.
Semantics: From Paths to Rectangles

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is called

- **path rectangle** iff $S_0 \subseteq \mathcal{S}$ and $M_i \subseteq \Omega$.
- **measurable path rectangle** iff $S_0 \in \mathcal{F}_\mathcal{S}$ and $M_i \in \mathcal{F}$.
Semantics: From Paths to Rectangles

Finite Measurable Path Rectangles

A set of path of length \( n \), represented as a Cartesian product

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- **path rectangle** iff \( S_0 \subseteq S \) and \( M_i \subseteq \Omega \).
- **measurable path rectangle** iff \( S_0 \in \mathcal{F}_S \) and \( M_i \in \mathcal{F} \).

Set of measurable path rectangles: \( \mathcal{F}_S \times \mathcal{F}^n \)
Semantics: From Paths to Rectangles

**Finite Measurable Path Rectangles**

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Set of measurable path rectangles: $\mathcal{F}_S \times \mathcal{F}^n$

**Lemma**

*The class of finite disjoint unions of measurable rectangles is a field.*
Semantics: The Product $\sigma$–Field

Finite Product $\sigma$–Field over Measurable Path Rectangles

The smallest $\sigma$–field generated by **measurable path rectangles**:

$$\mathcal{F}_{\text{Paths}}^n := \sigma\left(\mathcal{F}_S \times \mathcal{F}^n\right) \quad \text{for } n \geq 0.$$
Semantics: The Product $\sigma$–Field

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Example

Let $S = \{s_0, s_1\}$, Act = $\{a, b\}$.

- $\{s_0\} \times \{a, b\} \times (0, 0.2] \cup [1.2, 2] \times \{s_0, s_1\}$ is a measurable rectangle.

Elements: $s_0 \xrightarrow{a,0.1} s_0$, $s_0 \xrightarrow{a,0.1001} s_0$, $s_0 \xrightarrow{b,\sqrt{2}} s_1$, etc.
Semantics: The Product $\sigma$–Field

Finite Product $\sigma$–Field over Measurable Path Rectangles

The smallest $\sigma$–field generated by measurable path rectangles:

$$\mathcal{F}_{Paths}^{n} := \sigma \left( \mathcal{F}_{S} \times \mathcal{F}^{n} \right)$$

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Let $\mathcal{S} = \{s_0, s_1\}$, $\text{Act} = \{a, b\}$.

- $\{s_0\} \times \{a, b\} \times (0, 0.2] \cup [1.2, 2] \times \{s_0, s_1\}$ is a measurable rectangle.
  Elements: $s_0 \xrightarrow{a,0.1} s_0$, $s_0 \xrightarrow{a,0.1001} s_0$, $s_0 \xrightarrow{b,\sqrt{2}} s_1$, etc.

- $\left( \{s_1\} \times \{a\} \times (0, 0.2] \times \{s_2\} \right) \cup \left( \{s_1\} \times \{a, b\} \times [0.3, 1] \times \{s_3\} \right) \in \mathcal{F}_{Paths}^{1}$
  Elements: $s_1 \xrightarrow{a,0.1} s_2$, $s_1 \xrightarrow{b,\frac{1}{3}} s_3$, $s_1 \xrightarrow{a,0.2} s_2$, $s_1 \xrightarrow{a,\frac{1}{3}} s_3$, etc.
Cylinder Set Construction

Let $C^n \subseteq Paths^n$ be a set of finite paths. Its induced cylinder is

$$C_n := \{ \pi \in Paths^\omega \mid \pi[0..n] \in C^n \}$$

$C^n$ is the cylinder base of $C_n$. 
Cylinder Set Construction

Let $C^n \subseteq Paths^n$ be a set of finite paths. Its induced **cylinder** is

$$C_n := \{ \pi \in Paths^\omega \mid \pi[0..n] \in C^n \}$$

$C^n$ is the **cylinder base** of $C_n$.

$C_n$ is a **measurable cylinder** iff $C^n \in \mathcal{F}Paths^n$. 
Cylinder Set Construction

Let \( C^n \subseteq Paths^n \) be a set of finite paths. Its induced cylinder is

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\( C^n \) is the cylinder base of \( C_n \).

\( C_n \) is a measurable cylinder iff \( C^n \in \mathcal{F}Paths^n \).

Properties of Cylinders

- Any cylinder \( C \) can be represented by a finite cylinder base.
Cylinder Set Construction

Let $C^n \subseteq \text{Paths}^n$ be a set of finite paths. Its induced cylinder is

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$C^n$ is the cylinder base of $C_n$.

$C_n$ is a measurable cylinder iff $C^n \in \mathcal{F}\text{Paths}^n$.

Properties of Cylinders

- Any cylinder $C$ can be represented by a finite cylinder base.
- If $m < n$ and $C^m = C^n \times \Omega^{n-m}$, then $C_m = C_n$. 
Definition ($\sigma$–Field generated by Measurable Cylinders)

The minimal $\sigma$–field generated by measurable cylinders is defined by

$$\mathcal{F}_{Paths}^\omega := \sigma\left(\mathcal{F}_S \times \mathcal{F}^{\infty}\right)$$

or equivalently

$$\mathcal{F}_{Paths}^\omega := \sigma\left(\bigcup_{i=0}^{\infty} \{C_n \mid C^n \in \mathcal{F}_{Paths^n}\}\right).$$

Finally: $(\text{Paths}^\omega, \mathcal{F}_{Paths}^\omega)$ is our measurable space.
Semantics: Combined Transition Probability

Product Measure on Combined Transitions
For history $\pi \in Paths^*$, three types of measure spaces are involved:

1. $(\mathcal{A}, \mathcal{F}_\mathcal{A}, \mathcal{D}(\pi))$
Semantics: Combined Transition Probability

Product Measure on Combined Transitions

For history $\pi \in Paths^*$, three types of measure spaces are involved:

1. $(\text{Act}, \mathcal{F}_\text{Act}, \mathcal{D}(\pi))$

2. $(\mathbb{R}_{\geq 0}, \mathcal{B}(\mathbb{R}_{\geq 0}), \mu_a)$ where

   $\mu_a$’s distribution: $F(x) = \int_0^x E(\pi \downarrow, a) \cdot e^{-E(\pi \downarrow, a)t} \, dt$
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For history $\pi \in Paths^*$, three types of measure spaces are involved:

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3. $(S, \mathcal{F}_S, \mathbf{P}(\pi \downarrow, a))$. 
Semantics: Combined Transition Probability

Product Measure on Combined Transitions
For history $\pi \in Paths^*$, three types of measure spaces are involved:

1. $(\text{Act}, \mathcal{F}_{\text{Act}}, D(\pi))$
2. $(\mathbb{R}_{\geq 0}, \mathcal{B}(\mathbb{R}_{\geq 0}), \mu_a)$ where $\mu_a$'s distribution: $F(x) = \int_0^x E(\pi \downarrow, a) \cdot e^{-E(\pi \downarrow, a)t} dt$
3. $(\mathcal{S}, \mathcal{F}_S, P(\pi \downarrow, a))$.

Definition (A Measure on Subsets of $\Omega$)
Let $\pi \in Paths^*$. Then

$$\mu_D(\pi, \cdot) : \mathcal{F} \rightarrow [0, 1] :$$

$$M \mapsto \int_{\text{Act}} D(\pi, da) \int_{\mathbb{R}_{\geq 0}} \eta_a(dt) \int_S \underbrace{I_M(a, t, s)} \cdot P(\pi \downarrow, a, ds).$$
Semantics: Combined Transition Probability

Example (Probability Measure of Rectangles)
Let $\pi \in Paths^*$. Let $A \times I \times S' \in \mathcal{F}$, $I$ an interval. Then

$$\mu_{\mathcal{D}}(\pi, A \times I \times S') = \sum_{a \in A} D(\pi, \{a\}) \cdot P(\pi \downarrow, a, S') \cdot \int_I E(\pi \downarrow, a) \cdot e^{-E(\pi \downarrow, a)t} dt$$

Intuition
Semantics: Combined Transition Probability

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Intuition

- $D(\pi, \{a\})$: probability to leave $\pi \downarrow$ via action $a$
Semantics: Combined Transition Probability

Example (Probability Measure of Rectangles)
Let $\pi \in Paths^*$. Let $A \times I \times S' \in \mathcal{F}$, $I$ an interval. Then

$$
\mu_{\mathcal{D}}(\pi, A \times I \times S') = \sum_{a \in A} \mathcal{D}(\pi, \{a\}) \cdot P(\pi \downarrow, a, S') \cdot \int_I E(\pi \downarrow, a) \cdot e^{-E(\pi \downarrow, a)t} dt
$$

Intuition
- $\mathcal{D}(\pi, \{a\})$: probability to leave $\pi \downarrow$ via action $a$
- $P(\pi \downarrow, a, S')$: probability for $a$–successor in $S'$
Semantics: Combined Transition Probability

Example (Probability Measure of Rectangles)
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Intuition
- $D(\pi, \{a\})$: probability to leave $\pi \downarrow$ via action $a$
- $P(\pi \downarrow, a, S')$: probability for $a$–successor in $S'$
- $\int_I E(\pi \downarrow, a) \cdot e^{-E(\pi \downarrow, a)t} dt$: probability to leave $\pi \downarrow$ within $I$. 
Lemma (Measurability of $\mu_D$)

For fixed $M \in \mathcal{F}$ and finite path–length $n$:

$$\mu_D(\cdot, M) : (Paths^n, \mathcal{F}_{Paths^n}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

Measurability necessary for Lebesgue–integration.
Semantics: Probability Measures on $\mathcal{F}Paths^*$

Definition (Probability Measures on $\mathcal{F}Paths^*$)

Let $(\mathcal{S}, \text{Act}, R, AP, L)$ be a CTMDP, $\alpha$ an initial distribution and $\mathcal{D}$ a THR scheduler.
Semantics: Probability Measures on $\mathcal{F}Paths^*$

**Definition (Probability Measures on $\mathcal{F}Paths^*$)**

Let $(S, Act, R, AP, L)$ be a CTMDP, $\alpha$ an initial distribution and $D$ a THR scheduler. Define inductively

$$Pr_{\alpha, D}^0 : \mathcal{F}S \rightarrow [0, 1] : S \mapsto \sum_{s \in S} \alpha(s)$$

$$Pr_{\alpha, D}^n : \mathcal{F}Paths^n \rightarrow [0, 1] :$$

$$\Pi \mapsto \int_{Paths^{n-1}} Pr_{\alpha, D}^{n-1}(d\pi) \int_{\Omega} \left\{ \text{indicator} \right\}_{\Pi \circ m} \mu_D(\pi, dm).$$
Semantics: Probability Measures on $\mathcal{F} Paths^*$

Definition (Probability Measures on $\mathcal{F} Paths^*$)

Let $(S, Act, R, AP, L)$ be a CTMDP, $\alpha$ an initial distribution and $\mathcal{D}$ a $THR$ scheduler. Define inductively

$$
Pr^0_{\alpha, \mathcal{D}} : \mathcal{F} S \to [0, 1] : S \mapsto \sum_{s \in S} \alpha(s)
$$

$$
Pr^n_{\alpha, \mathcal{D}} : \mathcal{F} Paths^n \to [0, 1] :
\Pi \mapsto \int_{Paths^{n-1}} Pr^{n-1}_{\alpha, \mathcal{D}}(d\pi) \int_{\Omega} \{\text{indicator} \}
\times \mathcal{I}_{\Pi}(\pi \circ m) \mu_D(\pi, dm).
$$

Remarks:

- $m \in \Omega$ ranges over combined transitions.
Semantics: Probability Measures on $\mathcal{F}Paths^*$

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$$\Pi \mapsto \int_{Paths^{n-1}} Pr^{n-1}_{\alpha, \mathcal{D}}(d\pi) \int_{\Omega} \underbrace{I_{\Pi}(\pi \circ m)} \mu_{\mathcal{D}}(\pi, dm).$$

Remarks:

- $m \in \Omega$ ranges over combined transitions.
- If $\pi = s_0 \xrightarrow{a_0,t_0} s_1 \xrightarrow{a_1,t_1} \cdots \xrightarrow{a_{n-1},t_{n-1}} s_n$ and $m = (a, t, s)$, then

$$\pi \circ m := s_0 \xrightarrow{a_0,t_0} \cdots \xrightarrow{a_{n-1},t_{n-1}} s_n \xrightarrow{a,t} s.$$
Intuition: Probability of Rectangles

Example (Probability of Rectangles)
For measurable rectangle $\Pi \times M \in \mathcal{F}_{Paths^n}$, we obtain

$$Pr^n_{\alpha,D}(\Pi \times M) = \int_{\Pi} \mu_D(\pi, M) \ Pr^{n-1}_{\alpha,D}(d\pi)$$
**Intuition: Probability of Rectangles**

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For measurable rectangle $\Pi \times M \in \mathcal{F}_{Paths^n}$, we obtain

$$Pr_{\alpha,\mathcal{D}}^n(\Pi \times M) = \int_{\Pi} \mu_{\mathcal{D}}(\pi, M) Pr_{\alpha,\mathcal{D}}^{n-1}(d\pi)$$

where

$$\mu_{\mathcal{D}}(\pi, M) = \int_{\text{Act}} D(\pi, da) \int_{\mathbb{R}_{\geq 0}} \eta_a(dt) \int_S I_M(a, t, s) P(\pi \downarrow, a, ds).$$
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Intuition:

- $\Pi \in \mathcal{F}_{Paths^{n-1}}$ is a measurable set of paths,
Intuition: Probability of Rectangles

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For measurable rectangle $\Pi \times M \in \mathcal{F}_{\text{Paths}^n}$, we obtain

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Intuition:

- $\Pi \in \mathcal{F}_{\text{Paths}^{n-1}}$ is a measurable set of paths,
- $M \in \mathcal{F}$ is a set of combined transitions (e.g. $M = A \times I \times S'$).
A Probability Measure on $\mathcal{F}_{Paths^\omega}$

Any measurable cylinder $C$ can be represented as

$$C = \{ \pi \in Paths^\omega \mid \pi[0..n] \in C^n \} \text{ for some } n \geq 0 \text{ and } C^n \in \mathcal{F}_{Paths^n}. $$

Define the probability measure on measurable cylinders:

$$Pr^\omega_{\alpha, D} : \mathcal{F}_{Paths^\omega} \to [0, 1] : C_n \mapsto Pr^n_{\alpha, D}(C^n)$$
Semantics: Probability of Measurable Cylinders

A Probability Measure on $\mathcal{F}_{Paths^\omega}$

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**Theorem (Ionescu–Tulcea)**

$Pr^\omega_{\alpha, D}$ is well-defined and unique.
Semantics: Probability of Measurable Cylinders

A Probability Measure on $\mathcal{F}_{Paths^\omega}$

Any measurable cylinder $C$ can be represented as

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Define the probability measure on measurable cylinders:

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Theorem (Ionescu–Tulcea)

$Pr^\omega_{\alpha,D}$ is well-defined and unique.

Finally: $(Paths^\omega, \mathcal{F}_{Paths^\omega}, Pr^\omega_{\alpha,D})$ is the desired probability space.
The Logic $nC\text{SL}$

nondeterministic Continuous Stochastic Logic ($nC\text{SL}$).
Definition (Zeno Path)

Let $\mathcal{C} = (\mathcal{S}, \text{Act}, \mathbb{R}, \text{AP}, \mathcal{L})$ be a CTMDP and

$$\pi = s_0 \xrightarrow{a_0,t_0} s_1 \xrightarrow{a_1,t_1} s_2 \xrightarrow{a_2,t_2} \cdots.$$

$\pi$ is a zeno path iff the sequence $\sum_{i=0}^{n} t_i$ is convergent.
Zeno Behaviour

Definition (Zeno Path)
Let $\mathcal{C} = (S, Act, R, AP, L)$ be a CTMDP and

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$\pi$ is a zeno path iff the sequence $\sum_{i=0}^{n} t_i$ is convergent.

Lemma (Converging Paths Lemma)

The probability measure of the set of converging paths is zero.
Zeno Behaviour

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Let $C = (S, Act, R, AP, L)$ be a CTMDP and

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$\pi$ is a zeno path iff the sequence $\sum_{i=0}^{n} t_i$ is convergent.

Lemma (Converging Paths Lemma)
The probability measure of the set of converging paths is zero.

Example (What is it good for?)
For any $\pi \in Paths^\omega$ and $t \in \mathbb{R}_{\geq 0}$, $\pi @ t$ is well-defined.
Two kinds of property specifications:
Syntax of \textit{nCSL}

Two kinds of property specifications:

Example (Transient State Measures)

Given an initial distribution, what is the possibility to reach an error state within the first $t$ time units?
Syntax of $nCSL$

Two kinds of property specifications:

Example (Transient State Measures)
Given an initial distribution, what is the possibility to reach an error state within the first $t$ time units?

Example (Long Run Average Behaviour)
Average time spent in a blocking state.
Syntax of $nCSL$

**Definition ($nCSL$ Formulae)**

For $a \in \text{AP}$, $p \in [0, 1]$ and $\subseteq \in \{<, \le, \ge, >\}$, $nCSL$ state–formulas are built according to the following context–free grammar:

$$
\Phi :: \ a \mid \neg \Phi \mid \Phi \land \Phi \mid \exists^p \varphi \mid L \subseteq^p \Phi
$$
Syntax of $nCSL$

**Definition (nCSL Formulae)**

For $a \in \text{AP}$, $p \in [0, 1]$ and $\leq \in \{<, \leq, \geq, >\}$, $nCSL$ state–formulas are built according to the following context–free grammar:

$$
\Phi ::= a \mid \neg \Phi \mid \Phi \land \Phi \mid \exists p \varphi \mid L^p \Phi
$$

For $I \subseteq \mathbb{R}$ a nonempty interval, $nCSL$ path–formulas are defined by

$$
\varphi ::= X^I \Phi \mid \Phi U^I \Phi
$$
Semantics of $nCSL$ I

Definition (Semantics of State Formulae)
Let $\mathcal{C} = (\mathcal{S}, \text{Act}, \mathbf{R}, \text{AP}, L)$. Define

\[
\begin{align*}
    s \models a & \iff a \in L(s) \\
    s \models \neg \Phi & \iff \text{not } s \models \Phi \\
    s \models \Phi \land \Psi & \iff s \models \Phi \text{ and } s \models \Psi
\end{align*}
\]
Semantics of $nCSL$: Path Formulae

Definition (Semantics of Path Formulae)
For time–interval $I \subseteq \mathbb{R}$ and state formulas $\Phi$ and $\Psi$, define:

$$
\pi \models X^I \Phi \iff \pi[1] \models \Phi \land \delta(\pi, 0) \in I
$$

$$
\pi \models \Phi \cup^I \Psi \iff \exists t \in I. \ (\pi \upharpoonright t \models \Psi \land (\forall t' \in [0, t). \ \pi \upharpoonright t' \models \Phi))
$$
Semantics of nCSL: Transient State Measures

Definition (Transient State Formulae)

For probability bound $p \in [0, 1]$, comparison operator $\equiv$ and path formula $\varphi$, the transient state semantics is given by

$$ s \models \exists^{\equiv p} \varphi \iff \exists D \in THR. \Pr_{\alpha, D_s}^{\omega} \{ \pi \in \text{Paths}^{\omega} | \pi \models \varphi \} \equiv p $$
Semantics of \( nCSL \): Transient State Measures

**Definition (Transient State Formulae)**

For probability bound \( p \in [0, 1] \), comparison operator \( \equiv \) and path formula \( \varphi \), the transient state semantics is given by

\[
s \models \exists^p \varphi \iff \exists D \in \text{THR. } Pr_{\alpha, D_s}^\omega \{ \pi \in \text{Paths}^\omega \mid \pi \models \varphi \} \equiv p
\]

**Lemma (Measurability of Satisfying Paths)**

*For arbitrary path formula \( \varphi \):*

\[
\{ \pi \in \text{Paths}^\omega \mid \pi \models \varphi \} \in \mathcal{F} \text{Paths}^\omega.
\]
Semantics of $nCSL$: Long Run Average Behaviour I

Preliminaries
For CTMDP $\mathcal{C} = (S, \text{Act}, R, \text{AP}, L)$, state $s$ and state–formula $\Phi$:

What is the average amount of time spent in $\Phi$–states?
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Definition (state indicator)
Let $S \subseteq S$ and $t \in \mathbb{R}_{\geq 0}$. Then

$$h_{S,t} : \text{Paths}^\omega \rightarrow \{0, 1\} : \pi \mapsto \begin{cases} 1 & \text{if } \pi@t \in S \\ 0 & \text{otherwise} \end{cases}$$
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Intuition: Does \( \pi \) occupy a state from \( S \) at time–point \( t \)?

Lemma

The function \( h_{S,t} \) is measurable relative to \( (Paths^\omega, \mathcal{F}_{Paths^\omega}) \).
Semantics of \( nCSL \): Long Run Average Behaviour II

Deduction of Long Run Average Behaviour

The fraction of time spent in \( S \)-states on path \( \pi \in Paths^\omega \):

\[
g_{S,t} : Paths^\omega \rightarrow [0, 1] : \pi \mapsto \frac{1}{t} \int_0^t h_{S,t'}(\pi) \, dt'.
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\( g_{S,t} \) is measurable in \((Paths^\omega, \mathcal{F}_{Paths^\omega})\) and a random variable.
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Lemma

\( g_{S,t} \) is measurable in \( (Paths^\omega, \mathcal{F}_{Paths^\omega}) \) and a random variable.

Definition (Expectation)

Take the expectation \( g_{S,t} \) over \( \pi \in Paths^\omega \):

\[
E(g_{S,t}) = \int_{Paths^\omega} g_{S,t}(\pi) \, Pr_{\alpha, D}(d\pi).
\]
Semantics of $nCSL$: Long Run Average Behaviour III

Definition (Long Run Average Formulae)

For fraction $p \in [0, 1]$, comparison operator $\sqsubseteq$ and state formula $\Phi$, the long–run average semantics is defined by:

$$s \models L^p \Phi \iff \forall D \in THR. \lim_{t \to \infty} \int_{Paths_\omega} \left( \frac{1}{t} \int_0^t h_{Sat(\Phi),t'}(\pi)dt' \right) dPr_{\alpha,s,D} \sqsubseteq p$$
Semantics of $nCSL$: Long Run Average Behaviour III

Definition (Long Run Average Formulae)
For fraction $p \in [0, 1]$, comparison operator $\equiv$ and state formula $\Phi$, the long–run average semantics is defined by:

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Example
For CTMDP $C$ with initial state $s$ where $r \in \text{AP}$ labels all reactive states. The property

“99% of the time, the system directly reacts on input”

can be checked by the following $nCSL$–formula:

$$s \models L^{>0.99} r$$
Ongoing Work

1. Complete measurability issues in \( nC\text{SL} \)-semantics
   Measurability of \( \mathcal{U}^I \) subformulas.

2. Are all \( nC\text{SL} \)-formulas preserved under strong bisimulation?
   Provide the proof.

3. Which \( nC\text{SL} \)-restrictions are preserved under strong simulation?
   Define strong simulation on CTMDP, find appropriate restriction of \( nC\text{SL} \)
Thank you for your attention!