Delayed Nondeterminism in Continuous-Time Markov Decision Processes

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Imagine you have to come home by 6 pm.

- On your way, you stop at a red traffic light.
- When it turns green, you have two choices:
  - turn left: 1 min; traffic jam probability $\frac{1}{2}$.
  - turn right: 5 min; traffic jam probability $\frac{1}{9}$.
- Expected delay in a traffic jam: 30 min.

Best strategy to meet your family’s deadline?
Continuous-Time Markov Decision Processes: An Example

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  - Expected delay in a traffic jam: 30 min.
- Best strategy to meet your family’s deadline?

**Aim:** Maximize the probability to come **home** in $t$ time units.
CTMDPs are an important model in
- stochastic control theory
- stochastic scheduling

CTMDPs provide the semantic basis for
- non-well-specified stochastic activity networks
- generalised stochastic Petri nets with confusion
- Markovian process algebras

In this talk:
1. Introduction of CTMDPs.
2. Schedulers that resolve the nondeterminism.
3. Probability measures.
4. Delaying nondeterminism.
5. Results and future work.

Why Continuous-Time Markov Decision Processes?

1. CTMDPs are an important model in
   - stochastic control theory
   - stochastic scheduling
   [Qiu et al.]
   [Feinberg et al., Puterman]

2. CTMDPs provide the semantic basis for
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   - generalised stochastic Petri nets with confusion
   - Markovian process algebras
   [Sanders et al.]
   [Chiola et al.]
   [Hermanns et al., Hillston et al.]
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5. **Results** and future work.
Continuous Time Markov Decision Process

A tuple \((S, \text{Act}, R, \nu)\) is a CTMDP if \(S\) is a finite set of states and

- \(\text{Act} = \{\alpha, \beta, \gamma, \ldots\}\) is a finite set of actions and
- \(R : S \times \text{Act} \times S \rightarrow \mathbb{R}_{\geq 0}\) is a transition rate matrix such that
  - \(R(s, \alpha, s') = \lambda\) is the rate of a negative exponential distribution
    \[
    f_X(t) = \begin{cases} 
      \lambda \cdot e^{-\lambda \cdot t} & \text{if } t \geq 0 \\
      0 & \text{otherwise}
    \end{cases}
    \]
    and \(E[X] = \frac{1}{\lambda}\)
  - \(E(s, \alpha) = \sum_{s' \in S} R(s, \alpha, s')\) is the exit rate of \(s\) under \(\alpha\).

Example

1. Nondeterministically choose \(\beta \in \text{Act}(s_0)\).
2. Race between \(\delta\)-transitions in \(s_2\):
   - Mean delay: \(t_{\text{mean}} = 4\).
   - Probability to move to \(s_4\):
     \[
     R(s_2, \delta, s_4) = \frac{8}{9}.
     \]
Continuous Time Markov Decision Process

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  - \(E(s, \alpha) = \sum_{s' \in S} \mathbf{R}(s, \alpha, s')\) is the exit rate of \(s\) under \(\alpha\).

\[
f_X(t) = \begin{cases} 
\lambda \cdot e^{-\lambda \cdot t} & \text{if } t \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

such that \(\text{Act}(s) = \{\alpha \in \text{Act} \mid \exists s' \in S. \mathbf{R}(s, \alpha, s') > 0\} \neq \emptyset\) for all \(s \in S\).

Example

1. Nondeterministically choose \(\beta \in \text{Act}(s_0)\).
2. Race between \(\delta\)-transitions in \(s_2\):
   - Mean delay: \(\frac{1}{E(s_2, \delta)} = 4\).
   - Probability to move to \(s_4\): \(\frac{\mathbf{R}(s_2, \delta, s_4)}{E(s_4, \delta)} = \frac{8}{9}\).
Trajectories in CTMDPs

1. **Finite paths** of length $n \in \mathbb{N}$ are denoted $\pi = s_0 \xrightarrow{\alpha_0,t_0} \cdots \xrightarrow{\alpha_{n-1},t_{n-1}} s_n$.
   - $\pi \downarrow = s_n$ is the last state of $\pi$.
   - $\text{Paths}^n$ is the set of paths of length $n$ and

2. $\text{Paths}^\omega$ is the set of **infinite paths**.
Trajectories in CTMDPs

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   - \( \pi \downarrow = s_n \) is the last state of \( \pi \).
   - \( Paths^n \) is the set of paths of length \( n \) and
2. \( Paths^\omega \) is the set of infinite paths.

A **combined transition** \( m = (\alpha_n, t_n, s_{n+1}) \):
- \( \alpha_n \) is the action in state \( \pi \downarrow \) (chosen externally),
- \( t_n \) is the transition’s **firing time** and
- \( s_{n+1} \) the transition’s **successor** state.

\[ \Omega := Act \times \mathbb{R}_{\geq 0} \times S \] is the set of all combined transitions.
Constructing events in CTMDPs

Probability measures are defined on $\sigma$-fields:

$\mathcal{F}$ of sets of combined transitions:

$$\Omega := \text{Act} \times \mathbb{R}_{\geq 0} \times \mathcal{S}$$

$$\mathcal{F} := \sigma(\mathcal{F}_{\text{Act}} \times \mathcal{B}(\mathbb{R}_{\geq 0}) \times \mathcal{F}_S)$$

$\mathcal{F}_{\text{Paths}}$ of sets of paths of length $n$:

$$\mathcal{F}_{\text{Paths}} := \sigma(\{S_0 \times M_1 \times \cdots \times M_n | S_0 \in \mathcal{F}_S, M_i \in \mathcal{F}\})$$

$\mathcal{F}_{\text{Paths}}^\omega$ of sets of infinite paths:

Cylinder set construction:
- Any $C^n \in \mathcal{F}_{\text{Paths}}$ defines a cylinder base (of finite length).
- $C^n := \{\pi \in \text{Paths}^\omega | \pi[0..n] \in C^n\}$ is a cylinder (extension to infinity).

The $\sigma$-field $\mathcal{F}_{\text{Paths}}^\omega$ is then

$$\mathcal{F}_{\text{Paths}}^\omega := \sigma(\bigcup_{n=0}^{\infty} \{C^n | C^n \in \mathcal{F}_{\text{Paths}}\})$$
Constructing events in CTMDPs

Probability measures are defined on $\sigma$-fields:

1. $\mathcal{F}$ of sets of combined transitions:
   \[
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   \mathcal{F} := \sigma(\mathcal{F}_{\text{Act}} \times \mathcal{B}(\mathbb{R}_{\geq 0}) \times \mathcal{F}_{\mathcal{S}})
   \]

2. $\mathcal{F}_{\text{Paths}^n}$ of sets of paths of length $n$:
   \[
   \mathcal{F}_{\text{Paths}^n} := \sigma(\{S_0 \times M_1 \times \cdots \times M_n \mid S_0 \in \mathcal{F}_{\mathcal{S}}, M_i \in \mathcal{F}\})
   \]

3. $\mathcal{F}_{\text{Paths}^\infty}$ of sets of infinite paths:

   - Cylinder set construction:
     - Any $C^n \in \mathcal{F}_{\text{Paths}^n}$ defines a cylinder base (of finite length)
     - $C^n := \{s \in \text{Paths}^\infty \mid \pi[0..n] \in C^n\}$ is a cylinder (extension to infinity).

   The $\sigma$-field $\mathcal{F}_{\text{Paths}^\infty}$ is then
   \[
   \mathcal{F}_{\text{Paths}^\infty} := \sigma(\bigcup_{n=0}^{\infty} \{C^n \mid C^n \in \mathcal{F}_{\text{Paths}^n}\})
   \]
Constructing events in CTMDPs

Probability measures are defined on $\sigma$-fields:

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   \[ \Omega := \text{Act} \times \mathbb{R}_{\geq 0} \times \mathcal{S} \]
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2. $\mathcal{F}_{\text{Paths}^n}$ of sets of paths of length $n$:
   \[ \mathcal{F}_{\text{Paths}^n} := \sigma\left(\{S_0 \times M_1 \times \cdots \times M_n \mid S_0 \in \mathcal{F}_{\mathcal{S}}, M_i \in \mathcal{F}\}\right) \]

3. $\mathcal{F}_{\text{Paths}^\omega}$ of sets of infinite paths:
   Cylinder set construction:
   - Any $C^n \in \mathcal{F}_{\text{Paths}^n}$ defines a cylinder base (of finite length)
   - $C_n := \{\pi \in \text{Paths}^\omega \mid \pi[0..n] \in C^n\}$ is a cylinder (extension to infinity).

The $\sigma$-field $\mathcal{F}_{\text{Paths}^\omega}$ is then

\[ \mathcal{F}_{\text{Paths}^\omega} := \sigma\left(\bigcup_{n=0}^{\infty} \{C_n \mid C^n \in \mathcal{F}_{\text{Paths}^n}\}\right) \]

$\mathcal{B}(\mathbb{R}_{\geq 0})$: Borel $\sigma$-field for $\mathbb{R}_{\geq 0}$
Resolving nondeterminism: Assume state $s_n$ is hit after trajectory

$$\pi = s_0 \xrightarrow{\alpha_0, t_0} s_1 \xrightarrow{\alpha_1, t_1} s_2 \xrightarrow{\alpha_2, t_2} \ldots \xrightarrow{\alpha_{n-1}, t_{n-1}} s_n.$$ 

- Nondeterminism occurs in $s_n$ if $|\text{Act}(s_n)| > 1$.
- A scheduler resolves it and uniquely induces a stochastic process.
The probability of events

Resolving nondeterminism: Assume state $s_n$ is hit after trajectory

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- Nondeterminism occurs in $s_n$ if $|\text{Act}(s_n)| > 1$.
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A hierarchy of scheduler classes:

1. Generic measurable scheduler (GM):
   $$D : \text{Paths}^* \rightarrow \text{Distr}(\text{Act})$$

2. Total time positional scheduler (TTP):
   $$D : S \times \mathbb{R}_{\geq 0} \rightarrow \text{Distr}(\text{Act})$$

3. Time abstract hop counting scheduler (TAHOP):
   $$D : S \times \mathbb{N} \rightarrow \text{Distr}(\text{Act})$$

4. Time abstract positional scheduler (TAP):
   $$D : S \rightarrow \text{Distr}(\text{Act})$$
The probability of a single step $M \subseteq \mathcal{F}$

1. Enter state $s_n$ along trajectory
   $\pi = s_0 \xrightarrow{\alpha_0,t_0} s_1 \xrightarrow{\alpha_1,t_1} \cdots \xrightarrow{\alpha_{n-1},t_{n-1}} s_n$.

2. Continue in $s_n$ with a transition
   $(\alpha_n, t_n, s_{n+1}) \in M$

3. Measure probability of sets $M \subseteq \mathcal{F}$!
   **Example:** $M = \{\alpha_n\} \times [0, 1] \times \{s_{n+1}\}$.

**Probability measure** $\mu_D(\pi, \cdot) : \mathcal{F} \to [0, 1]$ on sets of combined transitions:

- Choose an action, wait and jump to successor state.

\[
\mu_D(\pi, M) := \int_{\text{Act}} D(\pi, d\alpha) \int_{\mathbb{R} \geq 0} \eta_E(\pi, \alpha)(dt) \int_{\mathcal{S}} I_M(\alpha, t, s') P(\pi, \alpha, ds')
\]

- Note: $\eta_E(\pi, \alpha)$ depends on scheduler $D$!
  **Therefore:** Scheduler cannot incorporate the sojourn time in state $\pi$.

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Delayed CTMDPs

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A generic probability measure on sets of paths

1. **Initial distribution** $\nu$: Probability to start in state $s$.

2. **$Pr^n_{\nu,D}$ on sets of finite paths**:
   Let $\nu \in \text{Distr}(S)$ and $D \in TTP$. Define inductively:
   
   $Pr^0_{\nu,D}(\Pi) := \sum_{s \in \Pi} \nu(s)$ \quad and for $n > 0$
   
   $Pr^n_{\nu,D}(\Pi) := \int_{\text{Paths}^{n-1}} Pr^{n-1}_{\nu,D}(d\pi) \int_{\Omega} 1_{\Pi}(\pi \circ m) \cdot \mu_D(\pi, dm)$

3. **$Pr^\omega_{\nu,D}$ on sets of infinite paths**:
   - A cylinder base is a measurable set $C^n \in \mathcal{F}_{\text{Paths}^n}$.
   - $C^n$ defines cylinder $\mathcal{C}_n = \{ \pi \in \text{Paths}^\omega \mid \pi[0..n] \in C^n \}$.
   - The probability of cylinder $\mathcal{C}_n$ is that of its base $C^n$:
     
     $Pr^\omega_{\nu,D}(\mathcal{C}_n) = Pr^n_{\nu,D}(C^n)$.
     
     This extends to $\mathcal{F}_{\text{Paths}^\omega}$ by Ionescu-Tulcea.
A generic probability measure on sets of paths

1. **Initial distribution** \( \nu \): Probability to start in state \( s \).

2. **\( \Pr_{\nu,D}^n \) on sets of finite paths:**
   Let \( \nu \in \text{Distr}(S) \) and \( D \in \text{TTP} \). Define inductively:
   \[
   \Pr_{\nu,D}^0(\Pi) := \sum_{s \in \Pi} \nu(s) \quad \text{and for } n > 0
   \]
   \[
   \Pr_{\nu,D}^n(\Pi) := \int_{\text{Paths}_{n-1}^\omega} \Pr_{\nu,D}^{n-1}(d\pi) \int_{\Omega} \mathbf{I}_\Pi(\pi \circ m) \mu_D(\pi, dm).
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3. **\( \Pr_{\nu,D}^{\omega} \) on sets of infinite paths:**
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Delaying the resolution of nondeterminism

- The semantics of a single step so far:

1. Scheduler decides upon entering $s_n$.
2. Sojourn time in $s_n$ depends on choice!

$$\int_{\text{Act}} D(\pi, d\alpha) \int_{\mathbb{R} \geq 0} \eta_E(\pi_\downarrow, \alpha)(dt) \int_{\mathcal{S}} I_{M}(\alpha, t, s') \cdot P(\pi_\downarrow, \alpha, ds')$$
Delaying the resolution of nondeterminism

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2. Sojourn time in \( s_n \) depends on choice!

- Idea to delay resolution of nondeterminism:
  **Schedule only when the current state is left!**
  Therefore: Dissolve dependency between
  - sojourn time in state \( s_n \) and
  - scheduler’s choice when entering \( s_n \).

\[ \int_{\mathbb{R}_{\geq 0}} \eta_{\lambda}(s_n)(dt) \int_{\text{Act}} D(\pi, t, d\alpha) \int_{S} \mathbf{I}_M(\alpha, t, s') \mathbf{P}(\pi_{\downarrow}, \alpha, ds') \]
Delaying the resolution of nondeterminism

- The semantics of a single step so far:

  1. Scheduler decides upon entering $s_n$.
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\int_{\mathbb{R} \geq 0} D(\pi, d\alpha) \int_{\mathbb{R} \geq 0} \eta_E(\pi_{\downarrow}, \alpha)(dt) \int_{s} I_M(\alpha, t, s') \ P(\pi_{\downarrow}, \alpha, ds')
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- Idea to delay resolution of nondeterminism:
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\int_{\mathbb{R} \geq 0} \eta_\lambda(s_n)(dt) \int_{\mathbb{R} \geq 0} D(\pi, t, d\alpha) \int_{s} I_M(\alpha, t, s') \ P(\pi_{\downarrow}, \alpha, ds')
\]
Local uniformity enables delayed scheduling

A CTMDP $\mathcal{C} = (\mathcal{S}, \text{Act}, \mathbb{R}, \nu)$ is **locally uniform** iff there exists $\lambda : \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}$ s.t.

$$\forall s \in \mathcal{S}. \forall \alpha \in \text{Act}(s). \lambda(s) = E(s, \alpha).$$

non-uniform CTMDP

![CTMDP Diagram]

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Local uniformity enables delayed scheduling

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$$\forall s \in \mathcal{S}. \forall \alpha \in \text{Act}(s). \quad \lambda(s) = E(s, \alpha).$$

**Local uniformization** yields $\text{unif}(\mathcal{C}) = (\overline{\mathcal{S}}, \text{Act}, \overline{\mathbf{R}}, \nu)$:

- $\overline{\mathcal{S}} = \mathcal{S} \cup \{s^\alpha \mid s \in \mathcal{S}, \alpha \in \text{Act} \text{ with } E(s, \alpha) < \lambda(s)\}$
- $\overline{\mathbf{R}}(s, \alpha, s') = \begin{cases} \mathbf{R}(s, \alpha, s') & \text{if } s, s' \in \mathcal{S} \\ \lambda(s) - E(s, \alpha) & \text{if } s \in \mathcal{S} \text{ and } s' = s^\alpha \\ 0 & \text{otherwise.} \end{cases}$
A hint towards correctness of local uniformization

non-uniform CTMDP

\[ E(s, \alpha) = \mu \text{ and } E(s, \beta) = \mu + \gamma \]
A hint towards correctness of local uniformization

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locally uniform CTMDP

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A hint towards correctness of local uniformization

**non-uniform CTMDP**

\[ E(s, \alpha) = \mu \text{ and } E(s, \beta) = \mu + \gamma \]

**Correctness:** If \( \alpha \) is chosen in \( s \), reachability of state \( u_i \) within \([0, t]\) is preserved:

\[ \frac{\mu_i}{\mu} \int_0^t \eta_{\mu}(dt) = \frac{\mu_i}{\mu + \gamma} \int_0^t \eta_{\mu+\gamma}(dt_1) + \frac{\mu}{\mu + \gamma} \int_0^t \eta_{\mu+\gamma}(dt_1) \frac{\mu_i}{\mu} \int_0^{t-t_1} \eta_{\mu}(dt_2) \]

where \( \eta_x = x \cdot e^{-x \cdot t} \) and \( \mu = \sum \mu_i \).

**locally uniform CTMDP**

\[ E(s, \alpha) = E(s, \beta) = \mu + \gamma \]
A hint towards correctness of local uniformization

non-uniform CTMDP

\[ E(s, \alpha) = \mu \] and \[ E(s, \beta) = \mu + \gamma \]

Correctness: If \( \alpha \) is chosen in \( s \), reachability of state \( u_i \) within \([0, t]\) is preserved:

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\]

where \( \eta_x = x \cdot e^{-x \cdot t} \) and \( \mu = \sum \mu_i \).

But: No nondeterminism considered yet!
A correspondence between paths in $C$ and $\text{unif}(C)$

The function $\text{merge} : \text{Paths}(\overline{C}) \rightarrow \text{Paths}(C)$ collapses copy-states $s^\alpha$:

$$\overline{\pi} = s_0 \xrightarrow{\beta,t_0} s'_0 \xrightarrow{\beta,t'_0} s_2 \xrightarrow{\delta,t_1} s_4$$

$$\text{merge}(\overline{\pi}) = s_0 \xrightarrow{\beta,t_0+t'_0} s_2 \xrightarrow{\delta,t_1} s_4.$$

The function $\text{extend} : \text{Paths}(C) \rightarrow \mathfrak{F}_{\text{Paths}(\overline{C})}$ is the inverse of $\text{merge}$. 
Resolving nondeterminism in $\text{unif}(C)$

Any CTMDP $C$ with $GM$ scheduler $D$ induces the measure $Pr_{\nu,D}^\omega$.

Question: How to mimic $D$’s behaviour on $\text{unif}(C)$ to obtain the same probability?
Resolving nondeterminism in \textit{unif} \((C)\)

Any CTMDP \(C\) with \(GM\) scheduler \(D\) induces the measure \(Pr^{\omega}_{\nu,D}\).

**Question:** How to mimic \(D\)'s behaviour on \textit{unif} \((C)\) to obtain the same probability?

**Definition (stutter scheduler)**

Let \(D\) be a \(GM\) scheduler on \(C\).

Define the stutter scheduler \(\overline{D}\) on \textit{unif} \((C)\):

\[
\overline{D}(\overline{\pi}) := \begin{cases} 
D(\pi) & \text{if } \overline{\pi} \downarrow \in S \land \text{merge}(\overline{\pi}) = \pi, \\
\{\alpha \mapsto 1\} & \text{if } \overline{\pi} \downarrow = s^\alpha.
\end{cases}
\]
Resolving nondeterminism in \( \text{unif}(C) \)

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**Definition (stutter scheduler)**

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\{\alpha \mapsto 1\} & \text{if } \pi \downarrow = s^\alpha.
\end{cases}
\]

**Note:** No choice in copy-state \( s_0^\beta \)
Soundness: From $\mathcal{C}$ to $\text{unif}(\mathcal{C})$

The construction of $\overline{\mathcal{D}}$ preserves all measures.

Proof sketch:

1. Uniformization is measure-preserving for measurable rectangles $\mathcal{C}^n$:

   $\Pr_{\mathcal{C},\mathcal{D}}^n(\mathcal{C}^n) = \Pr_{\mathcal{D}}^n(\text{extend}(\mathcal{C}^n))$

2. This extends to the field $\mathcal{G}_{Paths} = (\mathcal{F}_{S} \times \mathcal{F}_{Act} \times \mathcal{B}(\mathbb{R}_{\geq 0}))^n \times \mathcal{F}_{S}$.

3. Further we prove that

   $\mathcal{C} = \left\{ \Pi \in \mathcal{G}_{Paths} \mid \Pr_{\mathcal{C},\mathcal{D}}^n(\Pi) = \Pr_{\mathcal{D}}^n(\text{extend}(\Pi)) \right\}$

   is a monotone class.
Soundness: From $\mathcal{C}$ to $\text{unif}(\mathcal{C})$

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Proof sketch:

1. Uniformization is measure-preserving for measurable rectangles $C^n$:

   $$\Pr_{\nu,D}^n(C^n) = \overline{\Pr}_\nu^\omega(\text{extend}(C_n))$$

2. This extends to the field $\mathcal{G}_{\text{Paths}}^n = (\mathcal{F}_S \times \mathcal{F}_{\text{Act}} \times \mathcal{B}([\mathbb{R}_{\geq 0}])^n \times \mathcal{F}_S$.

3. Further we prove that

   $$\mathcal{C} = \left\{ \Pi \in \mathcal{F}_{\text{Paths}}^n(\mathcal{C}) \mid \Pr_{\nu,D}^n(\Pi) = \overline{\Pr}_\nu^\omega(\text{extend}(\Pi)) \right\}$$

   is a monotone class.
Soundness: From $\mathcal{C}$ to $\text{unif}(\mathcal{C})$

The construction of $\overline{\mathcal{D}}$ preserves all measures.

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The claim follows by applying the Monotone Class Theorem.
Completeness: From $\text{unif}(C)$ to $C$.

Main results:

1. For scheduler classes $\mathcal{G} \in \{TTP, TAP\}$:

$$\sup_{D \in \mathcal{G}(C)} Pr^{\omega}_{\nu,D}(\Pi) = \sup_{D' \in \mathcal{G}(C)} Pr^{\omega}_{\nu,D'}(\text{extend}(\Pi))$$

2. For the classes $\mathcal{G} \in \{TAHOP, TAH, TP\}$:

$$\sup_{D \in \mathcal{G}(C)} Pr^{\omega}_{\nu,D}(\Pi) \neq \sup_{D' \in \mathcal{G}(C)} Pr^{\omega}_{\nu,D'}(\text{extend}(\Pi))$$

3. Our main concern: Timed reachability analysis:
   - Previous results hold for arbitrary measures.
   - Reachability of states in $G$ in time $t$:

$$\sup_{D \in \text{TTP}(C)} Pr^{\omega}_{\nu,D}(\mathbf{[0,t]}^C) = \sup_{D \in \text{GM}(C)} Pr^{\omega}_{\nu,D}(\mathbf{[0,t]}^C).$$

Neuhausser, Stoelinga, Katoen (RWTH Aachen) Delayed CTMDPs FOSSACS 2009 16 / 18
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**Conjecture:** $GM$ and $TTH$ are also complete.

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The benefit of delaying nondeterminism

- Instead of early scheduling:

\[ \mu_D(\pi, M) = \int_{\mathbb{R}_{\geq 0}} D(\pi, d\alpha) \int_{\mathbb{R}_{\geq 0}} \eta_E(\pi \downarrow, \alpha) (dt) \int_S I_M(\alpha, t, s') P(\pi \downarrow, \alpha, d s'), \]

- local uniformity allows late scheduling:

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- What’s the benefit?
What is achieved:

We consider **locally uniform CTMDPs** and **late schedulers**:

1. They allow to delay the resolution of nondeterminism.
2. Late schedulers are strictly better than any early scheduler.
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**Future work:** **Timed reachability analysis.**

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Thank you for your attention!