Software Modeling and Verification
Lehrstuhl für Informatik 2
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Modeling Concurrent and Probabilistic Systems

Summer Term 09

– Series 9 –

Hand in until July 15 before the exercise class.

Exercise 1 (2 points)

Let X and Y be two probabilistic processes given by:

$$\begin{array}{rcl} X & = & \alpha.(X \oplus_{1/3} X') \\ X' & = & \beta.(X' \oplus_{1/7} \mathbf{0}) \\ \\ Y & = & \beta.(Y \oplus_{1/5} \gamma.Y) + \alpha.(Y \oplus_{1/4} \zeta.Y) \end{array}$$

Compute the asynchronous parallel composition $X||_{\{\alpha,\beta\}}Y$.

Solution Let $H = \{\alpha, \beta\}$.

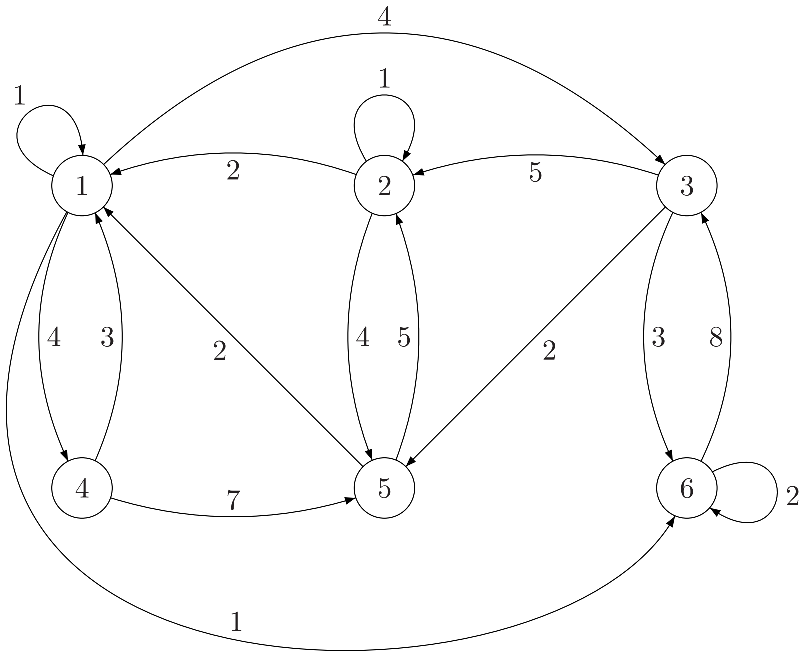
$$\begin{array}{rcl} X||_H\, Y & = & \alpha.([1/3 \times 1/4](X||_H\, Y) + [1/3 \times 3/4](X||_H\, \zeta.Y) \\ & & \quad +[2/3 \times 1/4](X'||_H\, Y) + [2/3 \times 3/4](X'||_H\, \zeta.Y)) \\ & = & \alpha.([1/12](X||_H\, Y) + [1/4](X||_H\, \zeta.Y) + [1/6](X'||_H\, Y) + [1/2](X'||_H\, \zeta.Y)) \end{array}$$

- $X||_H\, \zeta.Y = \zeta.(X||_H\, Y)$
- $X'||_H\, \zeta.Y = \zeta.(X'||_H\, Y)$
- $X'||_H\, Y = \beta.([1/35](X'||_H\, Y) + [4/35](X'||_H\, \gamma.Y)$
  $+[6/35](\mathbf{0}||_H\, Y) + [24/35](\mathbf{0}||_H\, \gamma.Y))$
  - $X'||_H\, \gamma.Y = \gamma.(X'||_H\, Y)$
  - $\mathbf{0}||_H\, \gamma.Y = \gamma.(\mathbf{0}||_H\, Y)$
  - $\mathbf{0}||_H\, Y = \mathbf{0}$

□

Exercise 2 (4 points)

Consider the CTMC $\mathcal{C}$, which is given by:

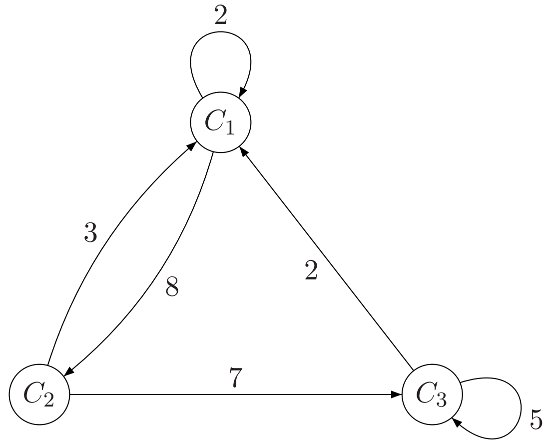


and its starting distribution $\underline{p}(0) = (0.1, 0.2, 0, 0, 0.4, 0.3)$.

1. Determine $\mathcal{C}/\sim_m$.
2. Determine the steady-state distribution of $\mathcal{C}$.

Solution

1. There are three equivalence classes in this CTMC. $C_1 = \{1, 6\}$, $C_2 = \{3, 4\}$ and $C_3 = \{2, 5\}$. They form the quotient CTMC as follows:



2. $\mathbf{R}= \begin{pmatrix} 1 & 0 & 4 & 4 & 0 & 1 \\ 2 & 1 & 0 & 0 & 4 & 0 \\ 0 & 5 & 0 & 0 & 2 & 3 \\ 3 & 0 & 0 & 0 & 7 & 0 \\ 2 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 2 \end{pmatrix}$, $\mathbf{Q}= \begin{pmatrix} -9 & 0 & 4 & 4 & 0 & 1 \\ 2 & -6 & 0 & 0 & 4 & 0 \\ 0 & 5 & -10 & 0 & 2 & 3 \\ 3 & 0 & 0 & -10 & 7 & 0 \\ 2 & 5 & 0 & 0 & -7 & 0 \\ 0 & 0 & 8 & 0 & 0 & -8 \end{pmatrix}$

The steady-state distribution $\underline{p} = (p_1, p_2, p_3, p_4, p_5, p_6)$ and $\underline{p} \cdot \mathbf{Q} = \underline{0}$. After solving a system of linear equations, we obtain $\underline{p} = (140/897, 3230/9867, 100/897, 56/897, 2776/9867, 55/897)$.

$\square$

Exercise 3 **(4 points)**

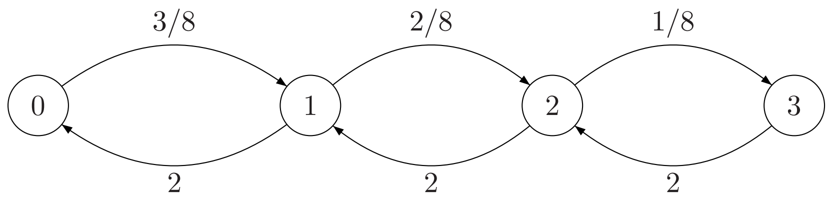
A professor supervises three Ph.D. students who all need quite a bit of advice. When any of these students visits the professor, the time to the next visit has an exponential distribution with a mean of 8 hours. The time for the professor to advise the students has a mean value of 1/2 hour.

All times in this problem have exponential distributions. Students visit the professor one at a time. If the professor is busy, the students wait outside his office. And the students are treated in a first-come-first-served manner.

1. Construct the CTMC that describes the situation.
2. At steady state what proportion of the professor's time does he have to himself (without students)?
3. Give the expression of computing the transient distribution of the CTMC at time $\hat{t}$. Suppose initially the professor is free. Use the uniformised CTMC.
4. If now comes a new Ph.D. student, who has the same arrival rate as the others, but requires a mean time of one hour with the professor. Do 1 again.

Solution

1. The situation can be modelled by the CTMC given below:



   where State 0 means there are no students meeting and the professor is free. State 1 means the professor is meeting one Ph.D. student and no other students are waiting. State 2 means that the professor is meeting one Ph.D. student and one other student is waiting. State 3 means that the professor is meeting one Ph.D. student and two other students are waiting.

2. The steady-state distribution of this CTMC is $(2048/2483, 384/2483, 48/2483, 3/2483)$. And at steady-state, $2048/2483$ of the professor's time belongs to himself.

3. The rate matrix $\mathbf{R}$, infinitesimal generator matrix $\mathbf{Q}$ are:

$$\mathbf{R} = \begin{pmatrix} 0 & 3/8 & 0 & 0 \\ 2 & 0 & 2/8 & 0 \\ 0 & 2 & 0 & 1/8 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} -3/8 & 3/8 & 0 & 0 \\ 2 & -9/4 & 1/4 & 0 \\ 0 & 2 & -17/8 & 1/8 \\ 0 & 0 & 2 & -2 \end{pmatrix}$$

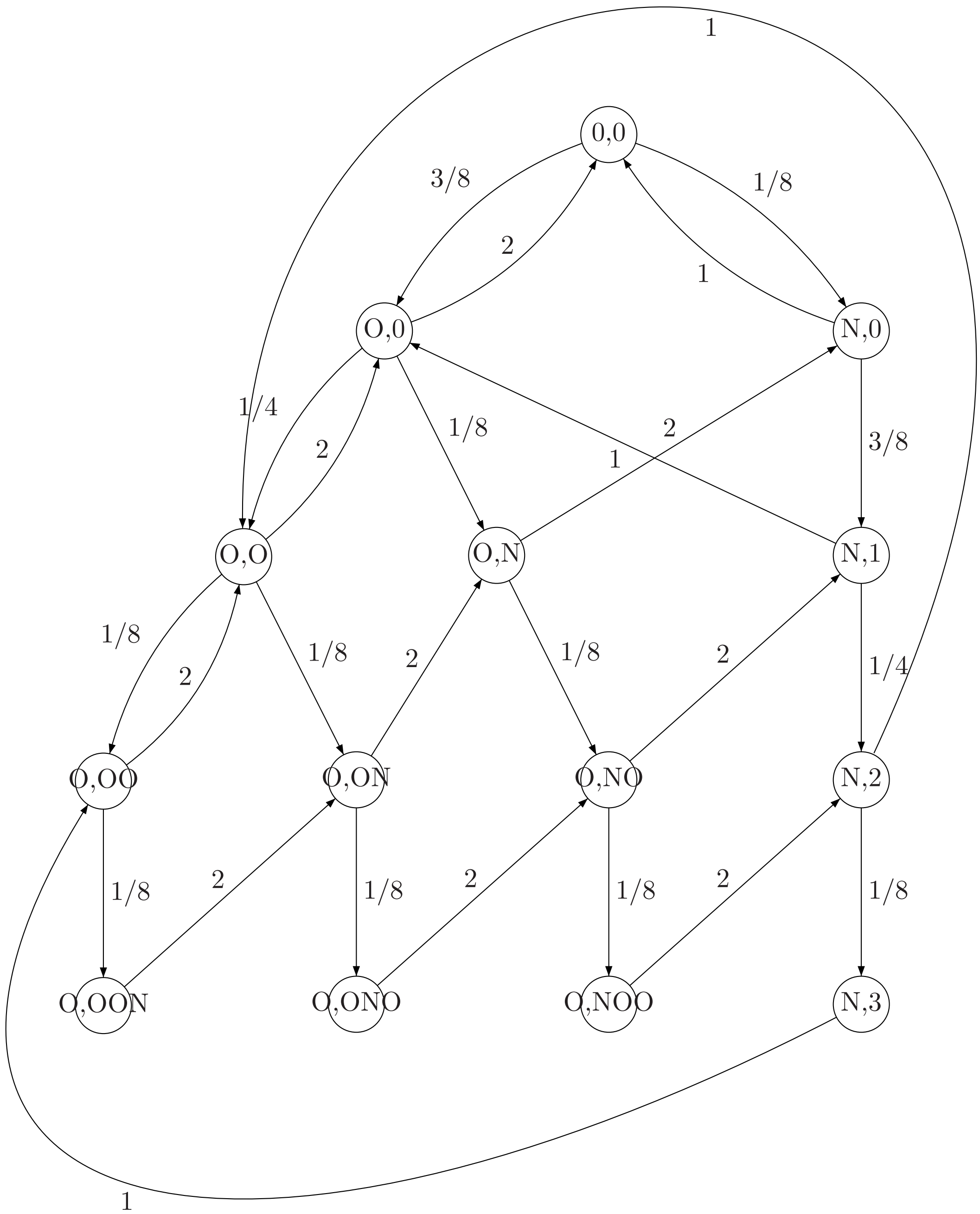
   And the transition probability matrix of the uniformised DTMC $\mathbf{U} = \mathbf{I} + \frac{\mathbf{Q}}{q}$ with $q = 9/4$ is:

$$\mathbf{U}= \begin{pmatrix} 5/6 & 1/6 & 0 & 0 \\ 8/9 & 0 & 1/9 & 0 \\ 0 & 8/9 & 1/18 & 1/18 \\ 0 & 0 & 8/9 & 1/9 \end{pmatrix}$$

The initial distribution $\underline{p}(0) = (1,0,0,0)$ and the transient distribution at time $\hat{t}$ is

$$\underline{p}(\hat{t}) = \underline{p}(0) \cdot e^{q(\mathbf{U}-\mathbf{I})\hat{t}} = \underline{p}(0) \cdot e^{-q\hat{t}} \cdot e^{q\hat{t}\mathbf{U}} = (1,0,0,0) \cdot e^{-9/4\hat{t}} \cdot e^{9/4\hat{t}\mathbf{U}}.$$

4. The CTMC for the new situation is:



where O means the first three old students and N means the new student. Each state is a pair. The first element of each pair denotes who is being met by the professor while the second element denotes the waiting list or the number of students waiting outside. For instance, state (0,0) means the professor is free, none students are being met or waiting. state (O,ONO) means the professor is meeting an old student, while the waiting list is ONO. (N,1) means that the professor is meeting the new student and one old student is waiting.

$\square$

Exercise 4 **(4 points)**

To prove: For CTMC $\mathcal{C}$ with $s_1, s_2 \in S$: $s_1 \approx_m s_2$ implies $s_1 \approx_p s_2$ in C's embedded DTMC $emb(\mathcal{C})$.

(Note that the definition of weak probabilistic bisimulation $\approx_p$ is given in Ex. 08.)

Solution Assume that R is a weak Markovian bisimulation on S. Let $s_1 R s_2$ and $B = [s_1]_R = [s_2]_R$. As $s_1 R s_2$, $\mathbf{R}(s_1, C) = \mathbf{R}(s_2, C)$ for all $C \in S/R$ with $C \neq B$. Hence

$$\mathbf{R}(s_1, S \setminus B) = \sum_{C \in S/R, C \neq B} \mathbf{R}(s_1, C) = \sum_{C \in S/R, C \neq B} \mathbf{R}(s_2, C) = \mathbf{R}(s_2, S \setminus B)$$

and

$$E(s_1) - \mathbf{R}(s_1, B) = \mathbf{R}(s_1, S \setminus B) = \mathbf{R}(s_2, S \setminus B) = E(s_2) - \mathbf{R}(s_2, B) \qquad (*)$$

If $\mathbf{P}(s_1, [s_1]_R) < 1$ and $\mathbf{P}(s_2, [s_2]_R) < 1$, we derive for any $C \in S/R$ with $C \neq B$:

$$\begin{aligned} \frac{\mathbf{P}(s_1, C)}{1 - \mathbf{P}(s_1, B)} &= \frac{E(s_1) \cdot \mathbf{P}(s_1, C)}{E(s_1) - E(s_1) \cdot \mathbf{P}(s_1, B)} = \frac{\mathbf{R}(s_1, C)}{E(s_1) - \mathbf{R}(s_1, B)} \\ &\stackrel{*}{=} \frac{\mathbf{R}(s_2, C)}{E(s_2) - \mathbf{R}(s_2, B)} = \frac{\mathbf{P}(s_2, C)}{1 - \mathbf{P}(s_2, B)} \end{aligned}$$

If $\mathbf{P}(s_1, [s_1]_R) = 1$ and $\mathbf{P}(s_2, [s_2]_R) = 1$, it means that s_1 cannot reach a state outside $[s_1]_R$, neither can s_2.

$\square$