RHEINISCH-WESTFÄLISCHE TECHNISCHE HOCHSCHULE AACHEN

Hybrid Systems Group
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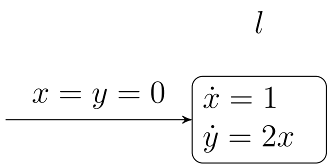
Modeling and Analysis of Hybrid Systems - SS 2009

Series 3

Exercise 1

Let be given the following simple hybrid automaton H:



Compute a polyhedral approximation for the flow pipe in l with initial set $X_0 = \{(0,0)^T\}$ and final time $t_f = 2$ in two segments. That means, compute

$$\hat{\mathcal{R}}_{[0,2]}(X_0) = \hat{\mathcal{R}}_{[0,1]}(X_0) \cup \hat{\mathcal{R}}_{[1,2]}(X_0).$$

6 points

Solution:

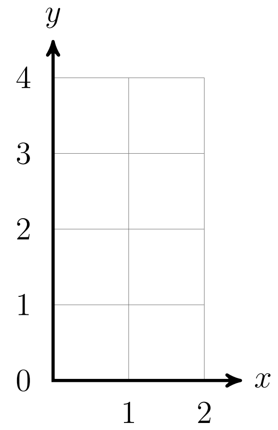
Note that we can compute an analytic solution of the differential equations. The function specifying the evolution of x is

$$\begin{aligned} f_x(x,y,t) &= x + \int_0^t 1\ dt \\ &= x + [1]_0^t \\ &= x + t. \end{aligned}$$

For the evolution of y we get:

$$\begin{aligned} f_y(x,y,t) &= y + \int_0^t 2 f_x(x,y,t)\ dt \\ &= y + \int_0^t 2(x+t)\ dt \\ &= y + \int_0^t 2x + 2t\ dt \\ &= y + [2xt + t^2]_0^t \\ &= y + 2xt + t^2. \end{aligned}$$

The flow pipe starting from X_0, i.e., from the point $(0,0)$ at $t=0$ in the x-y-plane looks as follows:



Now we compute a polyhedral approximation for the flow pipe $f(x,y,t) = (x+t, y+2xt+t^2)$ with initial set $X_0 = \{(0,0)\}$ and final time $t_f = 2$ in two segments:

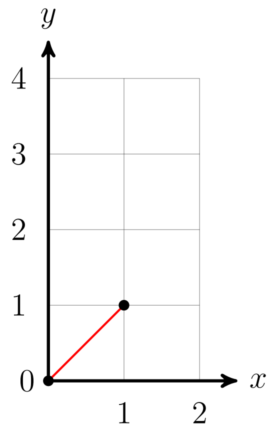
$$\hat{\mathcal{R}}_{[0,2]}(X_0) = \hat{\mathcal{R}}_{[0,1]}(X_0) \cup \hat{\mathcal{R}}_{[1,2]}(X_0).$$

1. Let's start with $\hat{\mathcal{R}}_{[0,1]}(X_0)$.
   - We have $V_0(X_0) = \{(0,0)^T\}$ and $V_1(X_0) = \{(1,1)^T\}$.
   - The convex hull of the vertices is defined by the constraints $x = y$ and $0 \le x \le 1$, i.e.,

     $$CH(V_0(X_0) \cup V_1(X_0)) = \{-x+y \le 0, x-y \le 0, -x \le 0, x \le 1\}.$$

     In matrix form,

     $$\underbrace{\begin{pmatrix} -1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} \le \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\vec{d}}.$$



- For the above matrix C we solve the optimization problem

$$\max_{x_0,t} c_i x(t, x_0) \quad s.t. \quad x_0 \in X_0, t \in [0,1]$$

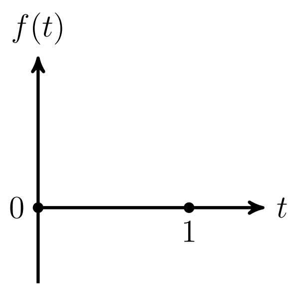
for $i = 1, \ldots, 4$, where c_i is the ith row of C. Since $X_0 = \{(0,0)^T\}$, the problem reduces to

$$\max_t c_i x(t,(0,0)^T) = c_i \begin{pmatrix} 0+t \\ 0+2\cdot 0\cdot t + t^2 \end{pmatrix} = c_i \begin{pmatrix} t \\ t^2 \end{pmatrix} \quad s.t. \quad t \in [0,1].$$

  - $i = 1$: maximize

$$c_1 x(t,(0,0)^T) = (-1,1) \begin{pmatrix} t \\ t^2 \end{pmatrix} = -t + t^2$$

with $t \in [0,1]$. The function $f(t) = -t + t^2$ in the interval $[0,1]$ behaves as follows:

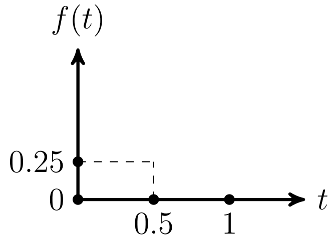


Since the first derivative of f is $-1 + 2t$ and the second derivative of f is 2, maxima exist only at the end points. The values at both endpoints $t = 0$ and $t = 1$ are maxima, and yield the same result. For $t^* = 1$ we get $d_1^* = c_1 x(t^*, x_0^*) = c_1 x(1,(0,0)^T) = (-1,1)(1,1)^T = 0$. (The other maximum at $t^* = 0$ would yield $d_1^* = c_1 x(t^*, x_0^*) = c_1 x(0,(0,0)^T) = (-1,1)(0,0)^T = 0$.)

– $i = 2$: maximize

$$c_2 x(t, (0,0)^T) = (1, -1) \begin{pmatrix} t \\ t^2 \end{pmatrix} = t - t^2$$

with $t \in [0,1]$. The function $f(t) = t - t^2$ in the interval $[0,1]$ behaves as follows:



The first derivative of $f(t) = t - t^2$ is $1 - 2t$, the second derivative is -2. Thus $1 - 2t = 0$, yielding $t = 0.5$, gives us the maximum value position. Using $t^* = 0.5$ we get $d_2^* = c_2 x(t^*, x_0^*) = c_1 x(0.5, (0,0)^T) = (1, -1)(0.5, 0.25)^T = 0.25$.

– $i = 3$: maximize

$$c_3 x(t, (0,0)^T) = (-1, 0) \begin{pmatrix} t \\ t^2 \end{pmatrix} = -t$$

with $t \in [0,1]$. The maximum is given by $t^* = 0$ and we get $d_3^* = c_3 x(t^*, x_0^*) = c_3 x(0, (0,0)^T) = (-1, 0)(0,0)^T = 0$.

– $i = 4$: maximize

$$c_4 x(t, (0,0)^T) = (1, 0) \begin{pmatrix} t \\ t^2 \end{pmatrix} = t$$

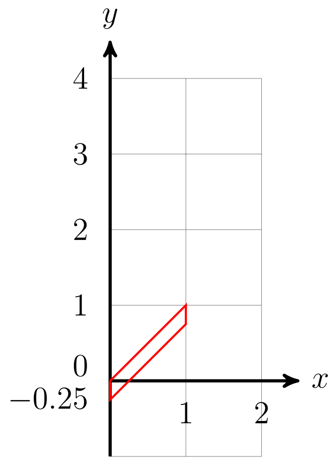
with $t \in [0,1]$. The maximum is given by $t^* = 1$ and we get $d_4^* = c_4 x(t^*, x_0^*) = c_1 x(1, (0,0)^T) = (1, 0)(1,1)^T = 1$.

Summarizing the four components gives $d^* = (0, 0.25, 0, 1)^T$.

The convex polyhedral approximation of the first flow pipe segment in matrix form:

$$\underbrace{\begin{pmatrix} -1 & 1 \\ 1 & -1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} \leq \underbrace{\begin{pmatrix} 0 \\ 0.25 \\ 0 \\ 1 \end{pmatrix}}_{\vec{d^*}}.$$

I.e., the approximation is determined by $y \leq x$, $y \geq x - 0.25$, and $0 \leq x \leq 1$:

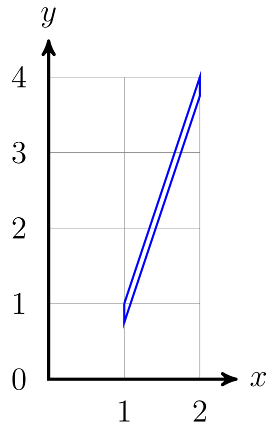


2. Computation of $\hat{\mathcal{R}}_{[1,2]}(X_0)$ is analogous.

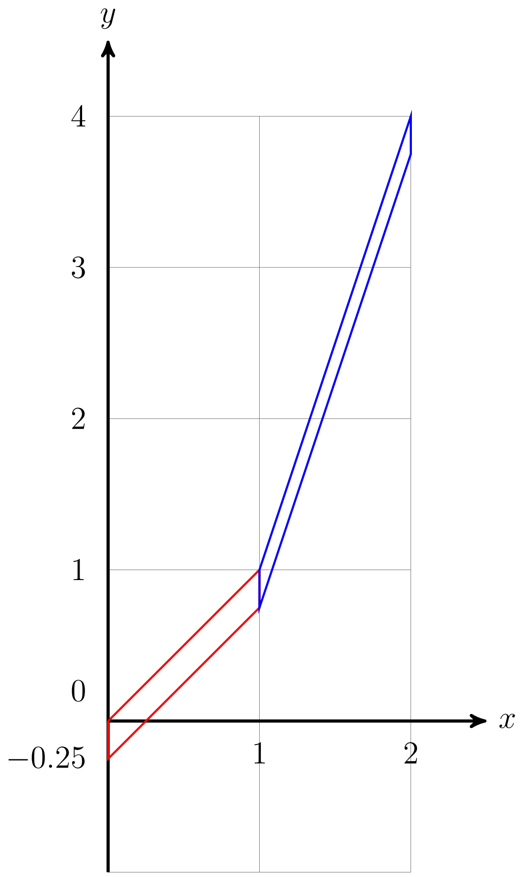
   The convex polyhedral approximation of the second flow pipe segment in matrix form:

$$\underbrace{\begin{pmatrix} -3 & 1 \\ 3 & -1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} \leq \underbrace{\begin{pmatrix} 2 \\ 2.25 \\ 0 \\ 1 \end{pmatrix}}_{\vec{d}^*}.$$

   I.e., the approximation is determined by $y \leq 3x+2$, $y \geq 3x-2.25$, and $0 \leq x \leq 1$:



Putting it together:



Hand in your solutions after the lecture on **Thursday,** .