Computing Approximations for the Reachable State Sets of Hybrid Systems

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Informatik 2 - Theory of Hybrid Systems
RWTH Aachen

SS09

Literatur

Alongkrit Chutinan and Bruce H. Krogh:
Computing Polyhedral Approximations to Flow Pipes for Dynamic Systems
In Proceedings of the 37rd IEEE Conference on Decision and Control, 1998

Olaf Stursberg and Bruce H. Krogh:
Efficient Representation and Computation of Reachable Sets for Hybrid Systems
Hybrid Systems: Computation and Control, LNCS 2623, pp. 482-497, 2003

We had a look at state set approximations by

- orthogonal polyhedra and
- oriented rectangular hull,

and at two basic operations

- testing for membership and
- intersection

on these.

Thus we can

- approximate state sets and
- compute with them.

How is all this used in the reachability analysis procedure?

General reachability procedure

Input: Set Init of initial states.

Algorithm:

$$
\begin{array}{l}
R^{\textsf{new}} := \textsf{Init};\\
R := \emptyset;\\
\textsf{while } (R^{\textsf{new}} \neq \emptyset)\{\\
\qquad R \quad := R \cup R^{\textsf{new}};\\
\qquad R^{\textsf{new}} := \boxed{\textsf{Reach}}(R^{\textsf{new}})\backslash R;\\
\}
\end{array}
$$

Output: Set R of reachable states.

What is "Reach"?

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For hybrid systems, independently of the exact definition of “Reach”, it will involve the following computations:

Given a state set R, compute

- the set of states reachable from R by a flow (i.e., time transisiton), and
- the set of states reachable from R by a jump (i.e., discrete transition).

Computing the jump successors, i.e., the flow pipe, of a set can be done with the operations we already introduced.

The harder part is computing the flow successors. So let’s start with that...

Approximating a flow pipe

Consider a dynamical system with state equation

$$\dot{x} = f(x(t)).$$

We assume f to be Lipschitz continuous so that for every initial state x_0 there is a unique solution $x(t, x_0)$ to the state equation.

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The set of reachable states at time t from a set of initial states X_0 is defined as

$$\mathcal{R}_t(X_0) = \{x_f \mid \exists x_0 \in X_0.\ x_f = x(t, x_0)\}.$$

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We describe a solution which approximates the flow pipe by a sequence of convex polytopes.

Definition (Convex polytope)

Let $POLY(C, d)$ denote the convex polytope defined by the pair $(C, d) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m$ according to

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- Given a finite set of points Γ, the convex hull $CH(\Gamma)$ of Γ is the smallest set (polytope) that contains Γ.

Problem statement for polyhedral approximation of flow pipes

Given

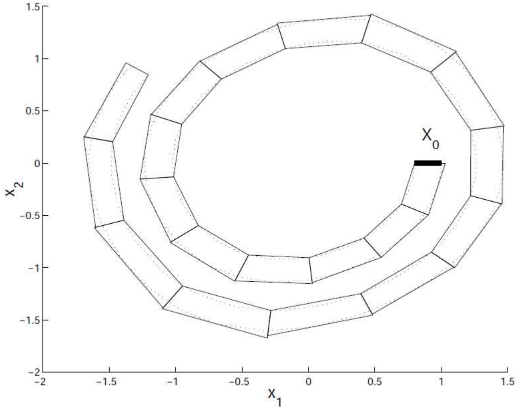
- a set X_0 of initial states which is a polytope, and
- a final time t_f,

compute a polyhedral approximation $\hat{\mathcal{R}}_{[0,t_f]}(X_0)$ to the flow pipe $\mathcal{R}_{[0,t_f]}(X_0)$ such that

$$\mathcal{R}_{[0,t_f]}(X_0) \subseteq \hat{\mathcal{R}}_{[0,t_f]}(X_0).$$

Flow pipe segmentation

Since a single convex polyhedra would strongly overapproximate the flow pipe, we compute a sequence of convex polyhedra, each approximating a flow pipe segment.



Segmented flow pipe approximation

Let the time interval $[0, t_f]$ be divided into $0 < N \in \mathbb{N}$ time segments

$$[0, t_1],\ [t_1, t_2],\ \ldots,\ [t_{N-1}, t_f]$$

with $t_i = i \cdot \frac{t_f}{N}$.

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We generate an approximation $\hat{\mathcal{R}}_{[t_1,t_2]}(X_0)$ for each flow pipe segment:

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$$\mathcal{R}_{[t_1,t_2]}(X_0) \subseteq \hat{\mathcal{R}}_{[t_1,t_2]}(X_0).$$

The complete flow pipe approximation is the union of the approximation of all N pipe segments:

$$\mathcal{R}_{[0,t_f]}(X_0) \subseteq \hat{\mathcal{R}}_{[0,t_f]}(X_0) = \bigcup_{k=1,\ldots,N} \hat{\mathcal{R}}_{[t_{k-1},t_k]}(X_0)$$

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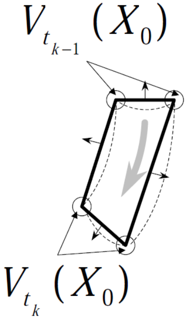

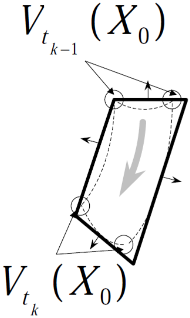

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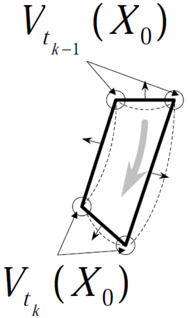

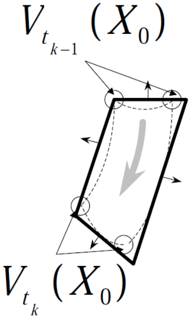

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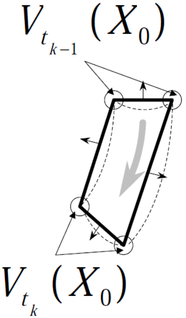
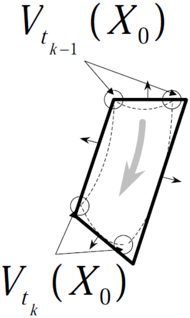
- Bloat hull: Enlarge the hull until it contains all points of the flow pipe segment.





1. Evolve vertices

To gain some geometrical information about the flow pipe segment, we begin with taking sample points at times t_{k-1} and t_k from the trajectories emanating from the vertices of X_0.

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In particular, we compute the sets $V_{t_{k-1}}(X_0)$ and $V_{t_k}(X_0)$ where

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Each point in the above sets can be obtained

- by analytic solution of the state equation and computing the value, or
- by simulation.

2. Determine hull

We use the evolved vertices in $V_{t_{k-1}}(X_0)$ and $V_{t_k}(X_0)$ to form a convex hull which serves as an initial approximation to the flow pipe segment $\mathcal{R}_{[t_{k-1},t_k]}(X_0)$, denoted by

$$\Phi_{[t_{k-1},t_k]}(X_0) = CH(V_{t_{k-1}}(X_0) \cup V_{t_k}(X_0)).$$

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Let (C_Φ, d_Φ) be the matrix-vector pair defining the convex hull, i.e.,

$$\Phi_{[t_{k-1},t_k]}(X_0) = POLY(C_\Phi, d_\Phi).$$

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- We use the normal vectors to the faces of this convex hull as a set of direction vectors to bloat the convex set until it contains the whole flow pipe segment.
- Given: $POLY(C_\Phi, d_\Phi)$.
- We want: $\mathcal{R}_{[t_{k-1},t_k]}(X_0) \subseteq POLY(C_\Phi, \boxed{\mathsf{d}})$.

3. Bloat hull

- We compute d as the solution to the following optimization problem:

$$\begin{aligned} \min_{d} \quad & volume[POLY(C_\Phi, d)] \\ s.t. \quad & \mathcal{R}_{[t_{k-1},t_k]}(X_0) \subseteq POLY(C_\Phi, d). \end{aligned} \tag{1}$$

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- The ith component d_i^* of the optimum d^* can be found by solving

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- Solution (x_0^*, t^*) to 3 $\rightarrow$

  Solution $x(t^*, x_0^*)$ to 2 $\rightarrow$

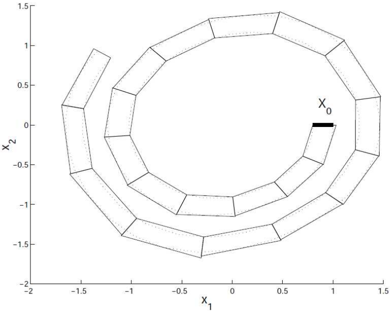
  Solution $d_i^* = c_i^T x(t^*, x_0^*)$ to 1.

Example

- Van der Pol equation:

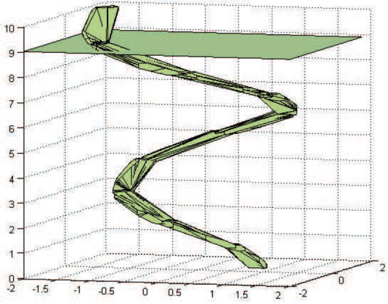
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -0.2(x_1^2 - 1)x_2 - x_1.\end{aligned}$$

- Intial set: $X_0 = \{(x_1, x_2) \mid 0.8 \leq x_1 \leq 1 \wedge x_2 = 0\}$.
- Time: $t_f = 10$.
- Segments: 20

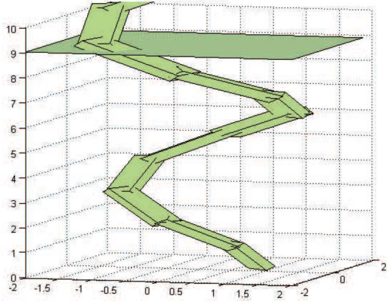


- Van der Pol equation with a third variable being a clock.
- Approximation

with convex polyhedra and



with oriented rectangular hull:



Partitioning the initial set

Var der Pol system with initial set $X_0 = \{(x_1, x_2) \mid 5 \leq x_1 \leq 45 \land x_2 = 0\}$.