# Reachable Set Representation and Computation for Hybrid Systems

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Informatik 2 - Theory of Hybrid Systems
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SS09

# Literatur

Oliver Bournez, Oded Maler, and Amir Pnueli:
Orthogonal Polyhedra: Representation and Computation
Hybrid Systems: Computation and Control, LNCS 1569, pp. 46-60, 1999

Olaf Stursberg and Bruce H. Krogh:
Efficient Representation and Computation of Reachable Sets for Hybrid Systems
Hybrid Systems: Computation and Control, LNCS 2623, pp. 482-497, 2003

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# Introduction

- There is a variety of methods for the verification of properties for hybrid systems.
- Most of them compute approximations for the set of reachable states in the continuous state space.
- Two approaches for computing states reachable by time steps (flows):
  1. Discretization partitions the state space into a finite number of subsets. An approximative evaluation of the continuous dynamics reveals which elements of the partition are reachable.
  2. Continuous dynamics can also be used to propagate the reachable set iteratively from the set of initial states.
- For the computation of states reachable by discrete steps (jumps) the conditions and effects must be evaluated.

# Computing reachability

## Continuous dynamics

Given a dynamical system defined by $\dot{x} = f(x)$, where $x$ takes values from $\mathbb{R}^d$, and given $P \subseteq \mathbb{R}^d$, calculate (or approximate) the set of points in $\mathbb{R}^d$ reached by trajectories (solutions) starting in $P$.

## Discrete steps

Given a discrete transition of a hybrid system with state space $\mathbb{R}^d$, and given $P \subseteq \mathbb{R}^d$, calculate (or approximate) the set of points in $\mathbb{R}^d$ reachable by taking the discrete transition starting in $P$.

# General reachability procedure

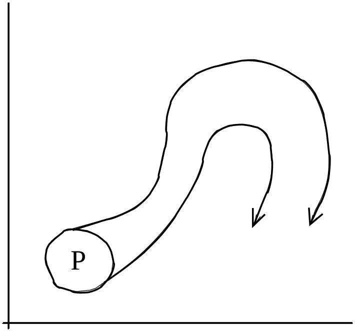
Input: Set Init of initial states.

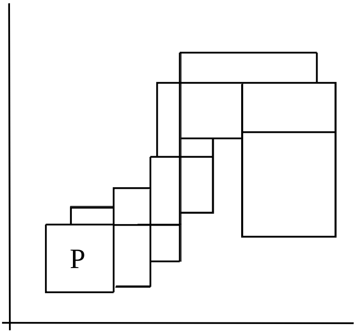
Algorithm:

$$
\begin{aligned}
&R^{\text{new}} := \text{Init};\\
&R := \emptyset;\\
&\text{while } (R^{\text{new}} \neq \emptyset)\{\\
&\qquad R \quad := R \cup R^{\text{new}};\\
&\qquad R^{\text{new}} := \text{Reach}(R^{\text{new}})\backslash R;\\
&\}
\end{aligned}
$$

Output: Set $R$ of reachable states.

# Reachability approximation for hybrid automata



# State set representation

- The geometry chosen to represent reachable sets has a crucial effect on the efficiency of the whole procedure.
- Usually, the more complex the geometry,
  1. the more costly is the storage of the sets,
  2. the more difficult it is to perform operations like union and intersection, and
  3. the more elaborate is the computation of new reachable sets, but
  4. the better the approximation of the set of reachable states.
- Choosing the geometry has to be a compromise between these impacts.

# Representation requirements

The geometry should allow efficient computation of the operations for

- membership relation,
- union,
- intersection,
- subtraction,
- test for emptiness.

# State set representation

Approaches:

- Polyhedra
- Orthogonal polyhedra
- Oriented rectangular hulls
- Zonotopes
- Ellipsoids

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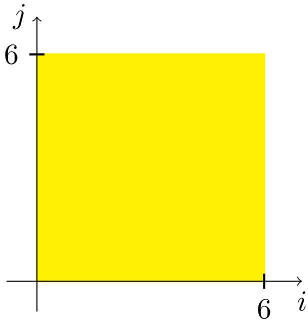
# The real domain

## Definition

- Domain: bounded subset $X = [0, m]^d \subseteq \mathbb{R}^d$ $(m \in \mathbb{N}_+)$ of the reals (can be extended to $X = \mathbb{R}^d_+$).
- Elements of $X$ are denoted by $\mathbf{x} = (x_1, \ldots, x_d)$, zero vector $\mathbf{0}$, unit vector $\mathbf{1}$.

$X = [0, 6]^2$



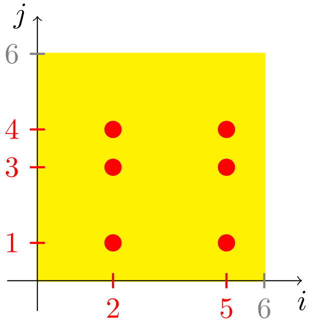
# Grids

## Definition

A $d$-dimensional grid associated with $X = [0, m]^d \subseteq \mathbb{R}^d$ $(m \in \mathbb{N}_+)$ is a product of $d$ subsets of $\{0, 1, \ldots, m-1\}$.

2-dimensional grid:

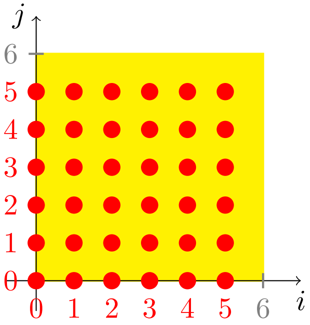
$$\{2, 5\} \times \{1, 3, 4\}$$



## Definition

The elementary grid associated with $X = [0,m]^d \subseteq \mathbb{R}^d$ $(m \in \mathbb{N}_+)$ is $\mathbf{G} = \{0, 1, \ldots, m-1\}^d \subseteq \mathbb{N}^d$.

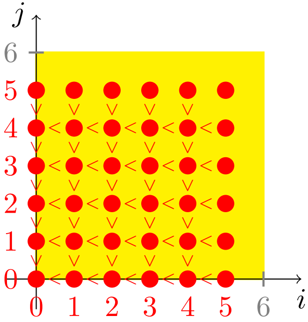
$$G = \{0, \ldots, 5\} \times \{0, \ldots, 5\}$$



# Grids

The grid admits a natural partial order with $(m-1,\ldots,m-1)$ on the top and $\mathbf{0}$ as bottom.

$$G = \{0,\ldots,5\} \times \{0,\ldots,5\}$$

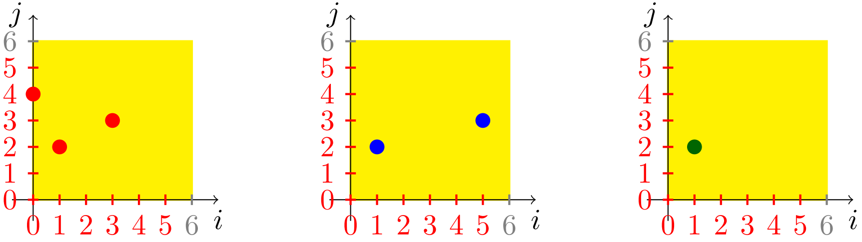


# Grids

The set of subsets of the elementary grid $\mathbf{G}$ forms a Boolean algebra $(2^{\mathbf{G}}, \cap, \cup, \sim)$ under the set-theoretic operations

- $A \cup B$
- $A \cap B$
- $\sim A = \mathsf{G} \backslash A$

for $A, B \subseteq \mathbf{G} \subset \mathbb{N}^d$.

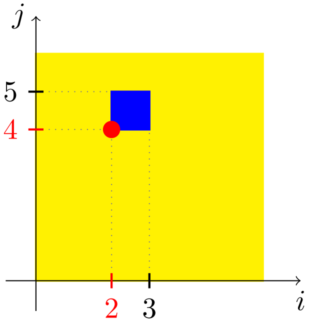



$\{(0,4),(1,2),(3,3)\} \quad \cap \quad \{(1,2),(5,3)\} \quad = \quad \{(1,2)\}$

## Definition (Elementary box)

- The elementary box associated with a grid point $\mathbf{x} = (x_1, \ldots, x_d)$ is $B(\mathbf{x}) = [x_1, x_1 + 1] \times \ldots, \times [x_d, x_d + 1]$.
- The point $\mathbf{x}$ is called the leftmost corner of $B(\mathbf{x})$.
- The set of elementary boxes is denoted by $\mathbf{B}$.

$$B((2,4)) = [2,3] \times [4,5]$$



## Definition (Orthogonal polyhedra)

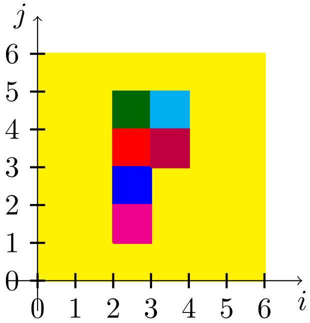
An orthogonal polyhedron $P$ is a union of elementary boxes, i.e., an element of $2^{\mathbf{B}}$.

$\{B((2,4))\} \cup \{B((3,4))\} \cup$

$\{B((2,3))\} \cup \{B((3,3))\} \cup$

$\{B((2,2))\} \cup$

$\{B((2,1))\}$



# Boolean algebra of orthogonal polyhedra

The set $2^{\mathbf{B}}$ of orthogonal polyhedra is closed under the following operations:

- $A \sqcup B = A \cup B$
- $A \sqcap B = cl(int(A) \cap int(B))$
- $\neg A = cl(\sim A)$

with

- $int$ the interior operator yielding the largest open set $int(A)$ contained in $A$, and
- $cl$ the topological closure operator yielding the smallest closed set $cl(A)$ containing $A$.

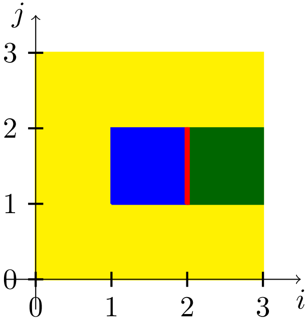
The set of orthogonal polyhedra forms a Boolean algebra $(2^{\mathbf{B}}, \sqcap, \sqcup, \neg)$.

$$A \sqcap B = cl(int(A) \cap int(B))$$

$([1,2] \times [1,2]) \sqcap ([2,3] \times [1,2]) =$

$cl(((1,2) \times (1,2)) \sqcap ((2,3) \times (1,2))) =$

$cl(\emptyset) = \emptyset$



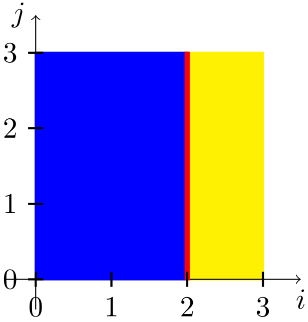

Note: $([1,2] \times [1,2]) \cap ([2,3] \times [1,2]) = [2,2] \times [1,2]$

$$\neg A = cl(\sim A)$$

$\neg([0,2] \times [0,3]) =$

$cl(\sim([0,2] \times [0,3])) =$

$cl((2,3] \times [0,3])) = [2,3] \times [0,3]$

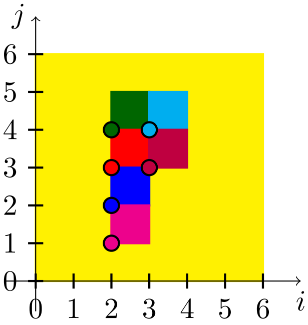



Note: $\sim([0,2] \times [0,3]) = (2,3] \times [0,3]$

# Connections

The bijection between $\mathbf{G}$ and $\mathbf{B}$ which associates every elementary box with its leftmost corner generates an isomorphism between $(2^{\mathbf{G}}, \cap, \cup, \sim)$ and $(2^{\mathbf{B}}, \sqcap, \sqcup, \neg)$.

Thus we can switch between point-based and box-based terminology according to what serves better the intuition.



# Color function

## Definition (Color function)

Let $P$ be an orthogonal polyhedron. The color function $c : X \rightarrow \{0, 1\}$ is defined by

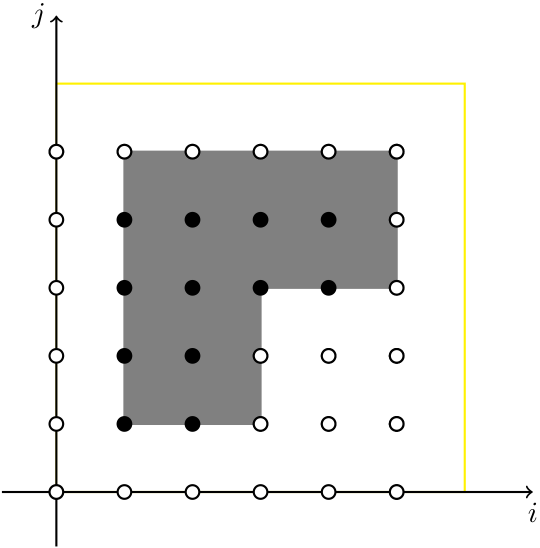
$$c(\mathbf{x}) = \begin{cases} 1 & \textit{if } \mathbf{x} \textit{ is a grid point and } B(\mathbf{x}) \subseteq P \\ 0 & \textit{otherwise} \end{cases}$$

for all $\mathbf{x} \in X$.

- If $c(\mathbf{x}) = 1$ we say that $\mathbf{x}$ is black and that $B(\mathbf{x})$ is full.
- If $c(\mathbf{x}) = 0$ we say that $\mathbf{x}$ is white and that $B(\mathbf{x})$ is empty.

Note that $c$ almost coincides with the characteristic function of $P$ as a subset of $X$. it differs from it only on right-boundary points.

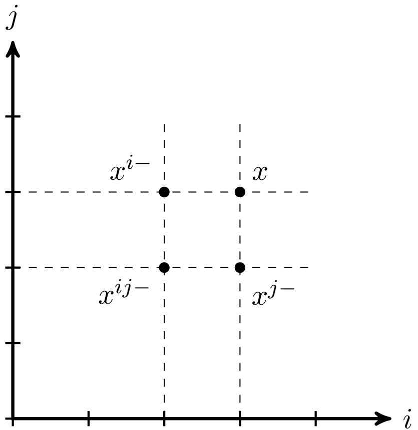
# Coloring



The following definitions capture the intuitive meaning of a facet and a vertex and, in particular, that the boundary of an orthogonal polyhedron is the union of its facets.

## Definition ($i$-predecessor)

The $i$-predecessor of a grid point $\mathbf{x} = (x_1, \ldots, x_d) \in X$ is $\mathbf{x}^{i-} = (x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_d)$. We use $\mathbf{x}^{ij-}$ to denote $(\mathbf{x}^{i-})^{j-}$. When $\mathbf{x}$ has no $i$-predecessor, we write $\bot$ for the predecessor value.



## Definition (Neighborhood)

The neighborhood of a grid point $\mathbf{x}$ is the set

$$\mathcal{N}(\mathbf{x}) = \{x_1 - 1, x_1\} \times \ldots \times \{x_d - 1, x_d\}$$

(the vertices of a box lying between $\mathbf{x} - \mathbf{1}$ and $\mathbf{x}$). For every $i$, $\mathcal{N}(\mathbf{x})$ can be partitioned into left and right $i$-neighborhoods

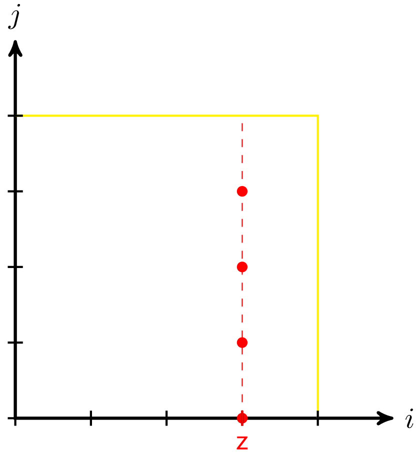
$$\mathcal{N}^{i-}(\mathbf{x}) = \{x_1 - 1, x_1\} \times \ldots \times \{x_i - 1\} \times \{x_d - 1, x_d\}$$

and

$$\mathcal{N}^{i}(\mathbf{x}) = \{x_1 - 1, x_1\} \times \ldots \times \{x_i\} \times \{x_d - 1, x_d\}.$$

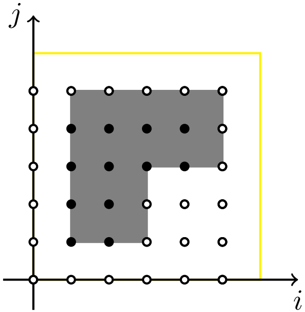
## Definition ($i$-hyperplane)

An $i$-hyperplane is a $(d-1)$-dimensional subset $H_{i,z}$ of $X$ consisting of all points $\mathbf{x}$ satisfying $x_i = z$.



Observations:

- Facets are $d-1$-dimensional polyhedra.
- As such, facets are subsets of $i$-hyperplanes.
- The coloring changes on facets.
- White vertices need special care (closure to the "right").



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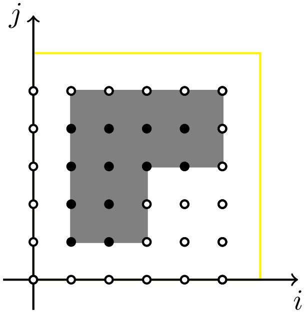



## Definition ($i$-facet)

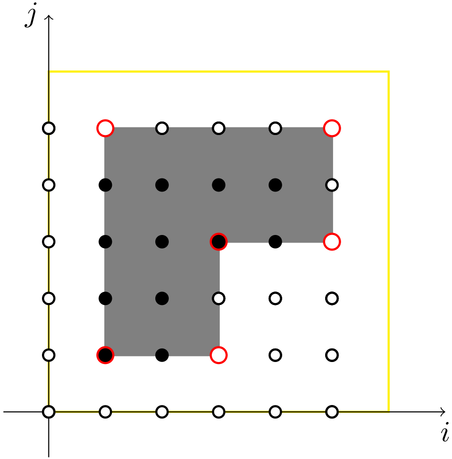
An $i$-facet of an orthogonal polyhedron $P$ with color function $c$ is

$$F_{i,z}(P) = cl\{\mathbf{x} \in H_{i,z} | c(\mathbf{x}) \neq c(\mathbf{x}^{i-})\}$$

for some integer $z \in [0, m)$.

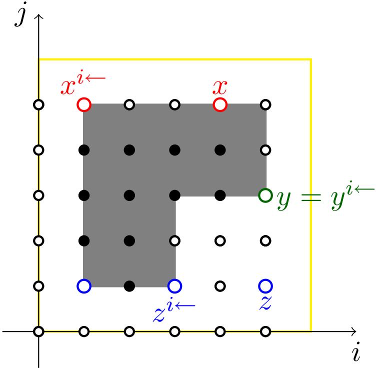
## Definition (Vertex)

A vertex is a non-empty intersection of $d$ distinct facets. The set of vertices of an orthogonal polyhedron $P$ is denoted by $V(P)$.



## Definition ($i$-vertex-predecessor)

- An $i$-vertex-predecessor of $\mathbf{x} = (x_1, \ldots, x_d) \in X$ is a vertex of the form $(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_d)$ for some integer $z \in [0, x_i]$. When $\mathbf{x}$ has no $i$-vertex-predecessor, we write $\perp$ for its value.
- The first $i$-vertex-predecessor of $\mathbf{x}$, denoted by $x^{i\leftarrow}$, is the one with the maximal $z$.



A representation scheme for $2^{\mathbf{B}}$ ($2^{\mathbf{G}}$) is a set $\mathcal{E}$ of syntactic objects such that there is a surjective function $\phi$ from $\mathcal{E}$ to $2^{\mathbf{B}}$, i.e., every syntactic object represents at most one polyhedron and every polyhedron has at least one corresponding object.

If $\phi$ is an injection we say that the representation is canonical, i.e., every polyhedron has a unique representation.

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- Explicit representation: an enumeration of the color values on every grid point, i.e., a $d$-dimensional zero-one array with $m^d$ entities.

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  - This representation is non-canonical.
- Vertex representation: consists of the set $\{(\mathbf{x}, c(\mathbf{x})) | \mathbf{x} \ \textit{is a vertex}\}$, i.e., the vertices of $P$ along with their color.
  - This representation is canonical.
  - The vertices alone is not a representation.
  - Not every set of points and colors is a valid representation of a polyhedron.

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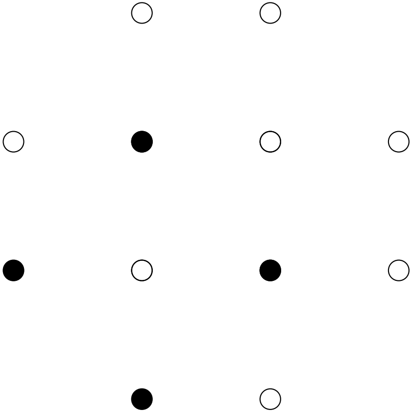
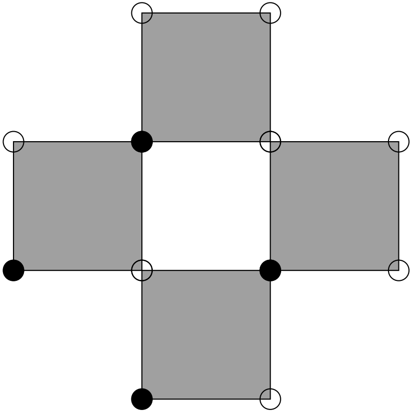
  - Not every set of points and colors is a valid representation of a polyhedron.
- Neighborhood representation: the colors of all the $2^d$ points in the neighborhoods of the vertices is attached as additional information.
- Extreme vertex representation: instead of maintaining all the neighborhood of each vertex, it suffices to keep only the *parity* of the number of black points in that neighborhood. In fact, it suffices to keep only vertices with odd parity.

# Vertex representation



# Vertex representation



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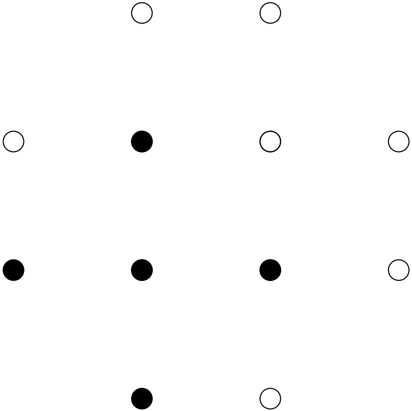

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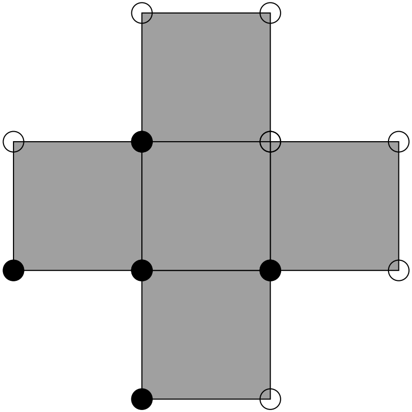
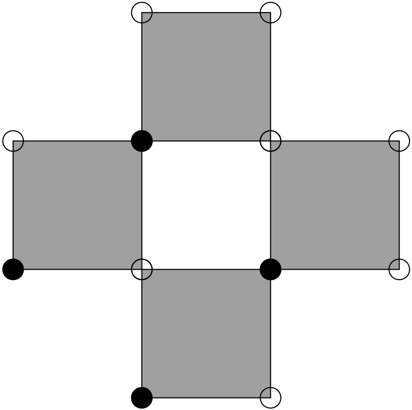
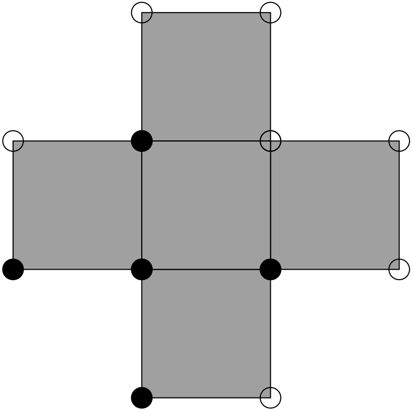


# Vertex representation





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# Membership problem

## The membership problem

Given a representation of a polyhedron $P$ and a grid point $\mathbf{x}$, determine $c(\mathbf{x})$, that is, whether $B(\mathbf{x}) \subseteq P$.

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# Membership problem for the vertex representation

## Observations

- A point $\mathbf{x}$ is on an $i$-facet iff

$$\exists \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}).\ c(\mathbf{x}'^{i-}) \neq c(\mathbf{x}').$$

- A point $\mathbf{x}$ is a vertex iff

$$\forall i \in \{1, \ldots, d\}.\ \exists \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}).\ c(\mathbf{x}'^{i-}) \neq c(\mathbf{x}').$$

- A point $\mathbf{x}$ is not a vertex iff

$$\exists i \in \{1, \ldots, d\}.\ \forall \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}).\ c(\mathbf{x}'^{i-}) = c(\mathbf{x}').$$

# Example

For $d = 2$ and $\mathbf{x} = (x_1, x_2)$ it means:

- $\mathbf{x}$ is on a $1$-facet iff

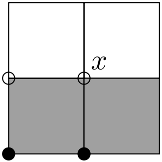
$$c(x_1 - 1, x_2 - 1) \neq c(x_1, x_2 - 1) \vee c(x_1 - 1, x_2) \neq c(x_1, x_2).$$

- $\mathbf{x}$ is on a $2$-facet iff

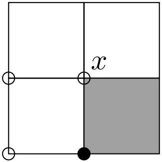
$$c(x_1 - 1, x_2 - 1) \neq c(x_1 - 1, x_2) \vee c(x_1, x_2 - 1) \neq c(x_1, x_2).$$

- $\mathbf{x}$ is a vertex iff both of the above hold.
- $\mathbf{x}$ is not a vertex iff one of the above does not hold.

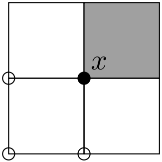
# Example



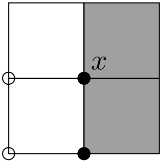

$$c(x_1, x_2 - 1) = c(x_1, x_2) \wedge$$
$$c(x_1 - 1, x_2 - 1) = c(x_1, x_2 - 1)$$




$$c(x_1 - 1, x_2 - 1) \neq c(x_1, x_2 - 1)$$




$$c(x_1, x_2 - 1) \neq c(x_1, x_2)$$




$$c(x_1, x_2 - 1) \neq c(x_1, x_2) \wedge$$
$$c(x_1 - 1, x_2 - 1) \neq c(x_1, x_2 - 1)$$

# Color computation

## Lemma (Color of a non-vertex)

*Let* $\mathbf{x}$ *be a non-vertex. Then there exists a direction* $j \in \{1, \ldots, d\}$ *such that*

$$\forall \mathbf{x}' \in \mathcal{N}^j(\mathbf{x}) \backslash \{\mathbf{x}\}.\ c(\mathbf{x}'^{j-}) = c(\mathbf{x}').$$

*Let* $j$ *be such a direction. Then* $c(\mathbf{x}) = c(\mathbf{x}^{j-})$*.*

# Color computation

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*Let* $j$ *be such a direction. Then* $c(\mathbf{x}) = c(\mathbf{x}^{j-})$*.*

*Proof*: A point $\mathbf{x}$ is not a vertex iff

$$\exists i \in \{1, \ldots, d\}.\ \forall \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}).\ c(\mathbf{x}'^{i-}) = c(\mathbf{x}').$$

Thus $j$ always exists. Let $i$ and $j$ two dimensions satisfying the above requirements.
Case 1: $j = i$: Straightforward
Case 2: $j \neq i$: For $i$ we have $c(\mathbf{x}^{i-}) = c(\mathbf{x})$ and $c(\mathbf{x}^{ij-}) = c(\mathbf{x}^{j-})$. For $j$ we have $c(\mathbf{x}^{ij-}) = c(\mathbf{x}^{j-})$.
Thus $c(\mathbf{x}) = c(\mathbf{x}^{j-})$.

# Complexity

Consequently we can calculate the color of a non-vertex $\mathbf{x}$ based on the color of all points in $\mathcal{N}(\mathbf{x}) - \{\mathbf{x}\}$: just find some $j$ satisfying the conditions of the above lemma and let $c(\mathbf{x}) = c(\mathbf{x}^{j-})$.

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*The membership problem for vertex representation can be solved in time $\mathcal{O}(n^d d 2^d)$ using space $\mathcal{O}(n^d)$.*

- We must recursively determine the color of at most $n^d$ grid points.
- For each of them we must check at most $d$ dimensions if they satisfy the condition of the lemma on the color of a non-vertex.
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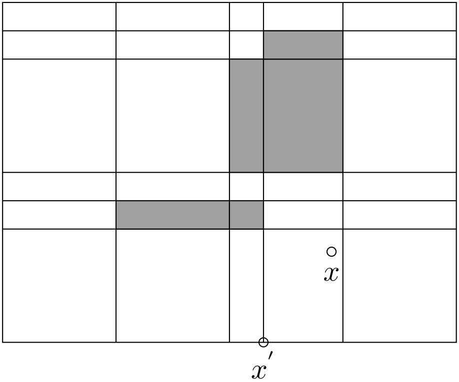
However, this algorithm is not very efficient, because in the worst-case one has to calculate the color of all the grid points between $\mathbf{0}$ and $\mathbf{x}$.

# Induced grid

We can improve it using the notion of an induced grid: let the $i$-scale of $P$ be the set of the $i$-coordinates of the vertices of $P$, and let the induced grid be the Cartesian product of its $i$-scales.



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- Every rectangle in the induced grid has a uniform color.
- Calculating the color of a point reduces to finding its closest "dominating" point on the induced grid and applying the algorithm to that grid in $\mathcal{O}(n^d d 2^d)$ time.

# Contents

# Membership problem for the neighborhood representation

We introduce an $\mathcal{O}(n \log n)$ membership algorithm for the neighborhood representation, based on successive projections of $P$ into polyhedra of smaller dimension.

# Membership problem for the neighborhood representation

## Definition ($i$-slice and $i$-section)

Let $P$ be an orthogonal polyhedron and $z$ an integer in $[0, m)$.

- The $i$-slice of $P$ at $z$ is the $d$-dimensional orthogonal polyhedron $J_{i,z}(P) = P \sqcap \{\mathbf{x} | z \leq x_i \leq z+1\}$.
- The $i$-section of $P$ at $z$ is the $(d-1)$-dimensional orthogonal polyhedron $\mathcal{J}_{i,z}(P) = J_{i,z}(P) \cap H_{i,z}$.

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Clearly, the membership of $\mathbf{x} = (x_1, \ldots, x_d)$ can be reduced into membership in $\mathcal{J}_{i,x_i}(P)$, which is a $(d-1)$-dimensional problem. By successively reducing dimensionality for every $i$ we obtain a point whose color is that of $\mathbf{x}$.

# Calculating the $i$-sections for the neighborhood representation

How can the main computational activity, the calculation of $i$-sections, be done using the neighborhood representation?

## Lemma (Vertex of a section)

*Let $P$ be an orthogonal polyhedron and let $P'$ be its $i$-section at $x_i = z$. A point $\mathbf{x}$ is a vertex of $P'$ iff $\mathbf{y} = \mathbf{x}^{i\leftarrow} \neq \bot$ and for every $j \neq i$ there exists $\mathbf{x}' \in \mathcal{N}^i(\mathbf{y}) \cap \mathcal{N}^j(\mathbf{y})$ such that $c(\mathbf{x}'^{j-}) \neq c(\mathbf{x}')$.*

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*Proof:* We prove: $\mathbf{x}$ is a vertex of $P'$ iff $\mathbf{y} = \mathbf{x}^{i\leftarrow} \neq \bot$ and for every $j \neq i$ there exists $\mathbf{x}' \in \mathcal{N}^i(\mathbf{y}) \cap \mathcal{N}^j(\mathbf{y})$ such that $c(\mathbf{x}'^{j-}) \neq c(\mathbf{x}')$.

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- Assume $\mathbf{x}$ is a vertex of $P'$. Then there is $\mathbf{y} = (x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_d)$ such that $c(N^i(\mathbf{y}) = c(N^i(\mathbf{x}))$ and $c(N^{i-}(\mathbf{y})) \neq c(N^i(\mathbf{y}))$ with $z$ maximal. Since $c(N^i(\mathbf{y}) = c(N^i(\mathbf{x}))$, $\mathbf{y}$ satisfies the condition as well. Since $c(N^{i-}(\mathbf{y})) \neq c(N^i(\mathbf{y}))$, $\mathbf{y}$ is a vertex of $P$. Since $z$ is maximal, $\mathbf{y} = \mathbf{x}^{i\leftarrow}$.

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- Assume conversely $\mathbf{y} = \mathbf{x}^{i\leftarrow}$ exists and it satisfies the condition. Then $c(N^i(\mathbf{x}) = c(N^i(\mathbf{y})$, because otherwise, by the above reasoning, there would be a vertex between $\mathbf{x}$ and $\mathbf{y}$. Hence $\mathbf{x}$ satisfies the condition.

## Theorem (Membership problem for neighborhood representation)

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- Hence it takes $\mathcal{O}(nd(\log n + 2^d))$ to get rid of one dimension.
- This is repeated $d$ times until $p$ is contrcted into a point.

A similar algorithm with the same complexity can be used to calculate the color of all the points in a neighborhood of $\mathbf{x}$.

The algorithm takes double slices ($d$-dimensional thick sections of width two) of $P$, and successively reduces $P$ into the neighborhood of $\mathbf{x}$.

This variation of the algorithm is used for doing Boolean operations.

# Contents

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- Instead of maintaining all the neighborhood of each vertex, it suffices to keep only the parity of the number of black points in that neighborhood – in fact it suffices to keep only vertices with odd parity.
- We use $parity(\mathbf{x})$ to denote the parity of the number of black points in $\mathcal{N}(\mathbf{x})$.
- A point $\mathbf{x}$ is said to be extreme if $parity(\mathbf{x}) = 1$.

# Membership problem for the extreme vertex representation

## Lemma

*An extreme point is a vertex.*

*Proof:* By induction on the dimension $d$. The base case $d = 1$ is immediate. For $d > 1$, choose $i \in \{1, \ldots, d\}$. Exactly one of $\mathcal{N}^{i-}(\mathbf{x})$ and $\mathcal{N}^{i}(\mathbf{x})$ contains an odd number of black points. Assume w.l.o.g. that it is $\mathcal{N}^{i}(\mathbf{x})$. By induction hypothesis $\mathbf{x}$ is a vertex in $\mathcal{J}_{i,x_i}(P)$. I.e., for every $j \neq i$ there exists $\mathbf{x}' \in \mathcal{N}^{j}(\mathbf{x})$ such that $c(x'^{j-}) \neq c(x')$. Since one cannot have $c(\mathbf{x}') = c(\mathbf{x}'^{i-})$ for all $\mathbf{x}' \in \mathcal{N}^{i}(\mathbf{x})$, $\mathbf{x}$ is a vertex of $P$.

The converse is not true, i.e., vertices need not be extreme.

- An extreme vertex representation consists in representing an orthogonal polyhedron by the set of its extreme vertices. (Additionally, the color of the origin is stored in a bit. From this information the colors of all extreme vertices can be inferred.)

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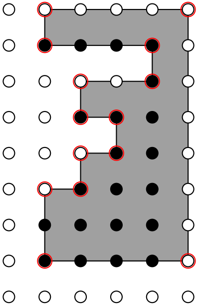

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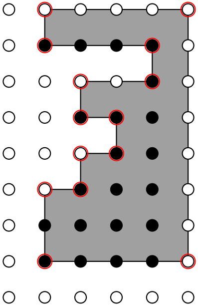

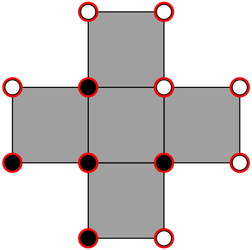

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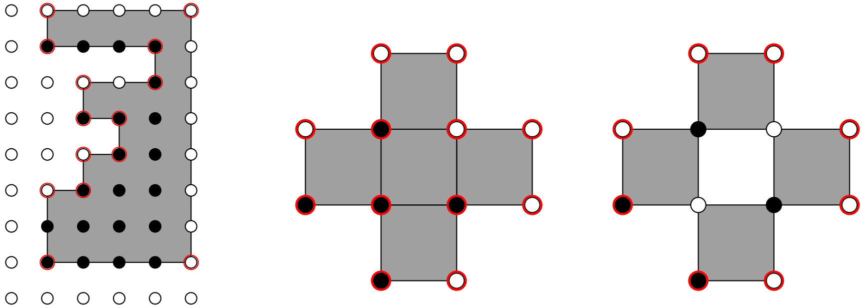

The membership problem is solved again by projection. For that we need again a rule to determine which points of an $i$-section are extreme vertices.

## Lemma (Extreme vertices of a section)

*Let $P$ be an orthogonal polyhedron and let $P' = \mathcal{J}_{i,z}(P)$. A point $\mathbf{x}$ is an extreme vertex of $P'$ iff it has an odd number of extreme $i$-vertex-predecessors.*

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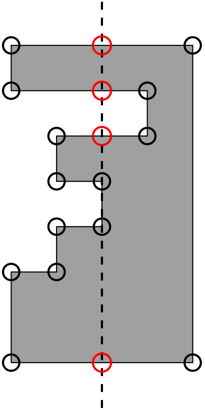

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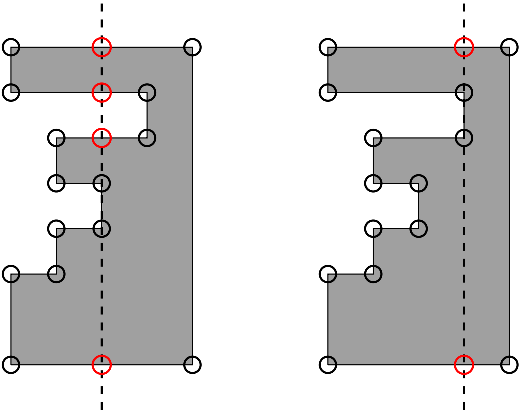

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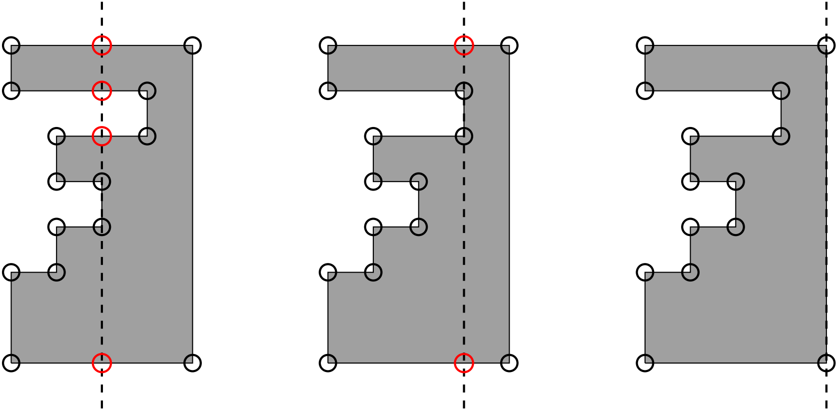

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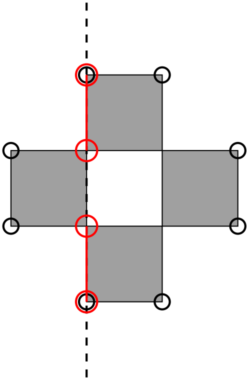

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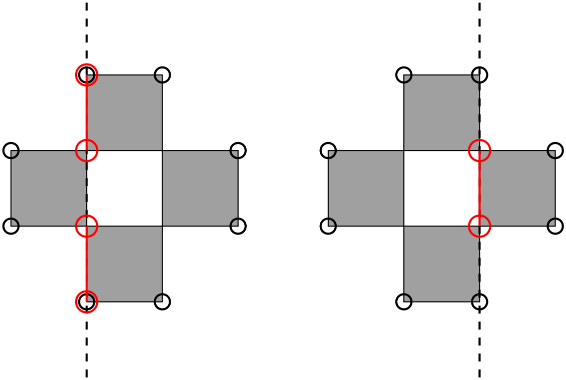

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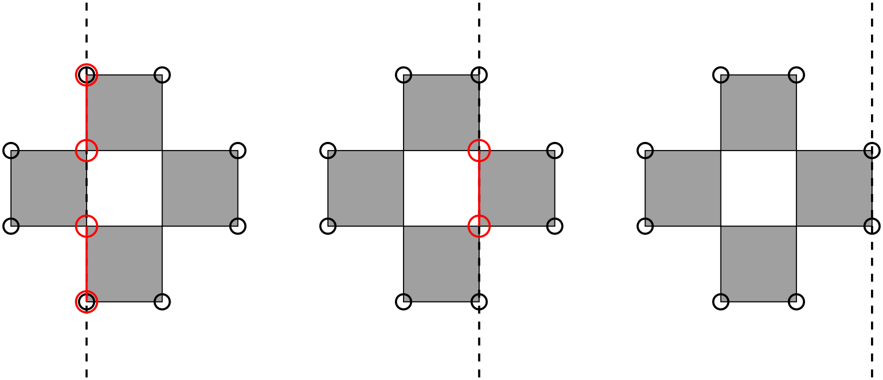

# Contents

# Intersection

We assume two polyhedra $P_1$ and $P_2$ with $n_1$ and $n_2$ vertices, respectively. After intersection some vertices disappear and some new vertices are created.

# Intersection

## Lemma

*A point* $\mathbf{x}$ *is a vertex of* $P_1 \cap P_2$ *only if for every dimension* $i$, $\mathbf{x}$ *is on an* $i$*-facet of* $P_1$ *or on an* $i$*-facet of* $P_2$.

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*Let* $\mathbf{x}$ *be a vertex of* $P_1 \cap P_2$ *which is not an original vertex.*
*Then there exists a vertex* $\mathbf{y}_1$ *of* $P_1$ *and a vertex* $\mathbf{y}_2$ *of* $P_2$ *such that* $\mathbf{x} = \max(\mathbf{y}_1, \mathbf{y}_2)$, *where* $\max$ *is applied componentwise.*

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*Conclusion:* the candidates for being vertices of $P_1 \cap P_2$ are restricted to:

$$V(P_1) \cup V(P_2) \cup \{\mathbf{x} | \exists \mathbf{y}_1 \in V(P_1).\ \exists \mathbf{y}_2 \in V(P_2).\ \mathbf{x} = \max(\mathbf{y}_1, \mathbf{y}_2)\}$$

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whose number is not greater then $n_1 + n_2 + n_1 n_2$.

# Intersection

# Intersection computation: Vertex representation

Computation of the intersection of two polyhedra $P_1$ and $P_2$:

# Intersection computation: Vertex representation

Computation of the intersection of two polyhedra $P_1$ and $P_2$:

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  - Use the vertex rules to determine, whether the point is a vertex of the intersection.

# Intersection example: Vertex representation

Vertex rule: A point $\mathbf{x}$ is a vertex iff

$$\forall i \in \{1, \ldots, d\}.\ \exists \mathbf{x}' \in \mathcal{N}^i(\mathbf{x}).\ c(\mathbf{x}'^{i-}) \neq c(\mathbf{x}').$$

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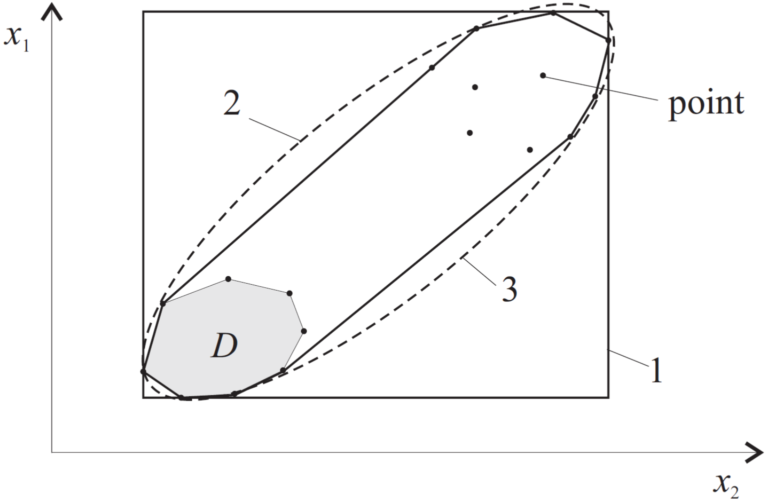
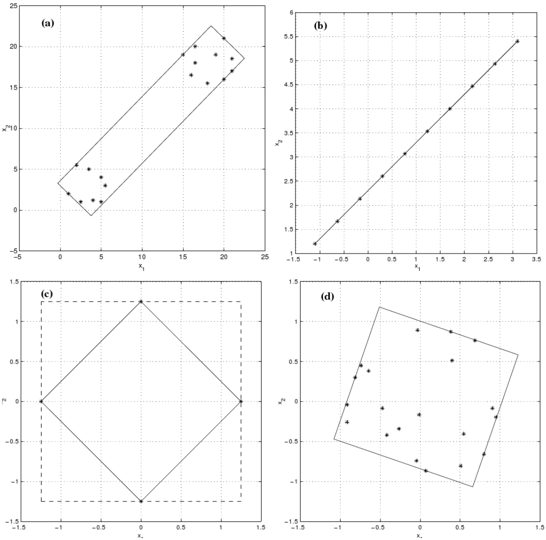
# Contents

# Motivation



# Oriented rectangular hull



# Principal component analysis

Principal component analysis (PCA)

- transforms some given data
- to a new coordinate system such that
- the greatest variance by any projection of the data comes to lie on the first coordinate (called the first principal component),
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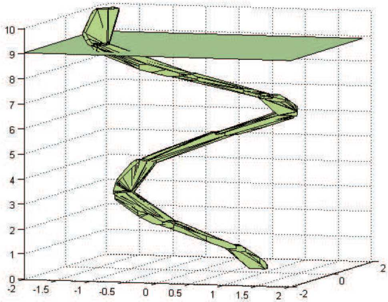
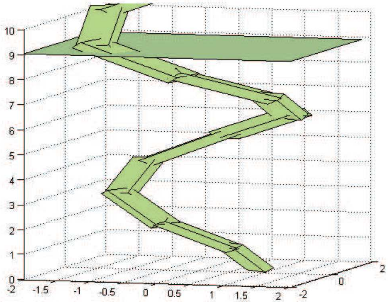
PCA involves the calculation of the eigenvalue decomposition of a data covariance matrix (or singular value decomposition of a data matrix), after mean centering the data for each attribute.

# Oriented rectangular hulls in reachability computation



Given a vector of sample points $X = (x^1, \ldots, x^p)$ with $x^i \in \mathbb{R}^n$, its arithmetic mean is

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In matrix form:

$$\bar{X} = (x^1, \ldots, x^p) = \begin{pmatrix} \bar{x}_1^1 & \cdot & \cdot & \cdot & \bar{x}_1^p \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \bar{x}_n^1 & \cdot & \cdot & \cdot & \bar{x}_n^p \end{pmatrix}.$$

# Example

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For

$$\bar{X} = (x^1, \ldots, x^p) = \begin{pmatrix} \bar{x}_1^1 & \cdot & \cdot & \cdot & \bar{x}_1^p \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \bar{x}_n^1 & \cdot & \cdot & \cdot & \bar{x}_n^p \end{pmatrix}$$

we define the sample covariance matrix

$$\text{Cov}(\bar{X}) = \begin{pmatrix} \text{Cov}(\bar{x}_1, \bar{x}_1) & \cdot & \cdot & \cdot & \text{Cov}(\bar{x}_1, \bar{x}_n) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \text{Cov}(\bar{x}_n, \bar{x}_1) & \cdot & \cdot & \cdot & \text{Cov}(\bar{x}_n, \bar{x}_n) \end{pmatrix}$$

with

$$\text{Cov}(\bar{x}_i, \bar{x}_j) = \frac{1}{p-1} \sum_{k=1}^{p} \bar{x}_i^k \cdot \bar{x}_j^k$$

for all $0 \leq i, j \leq n$.

# Example

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$$\bar{X} = \begin{pmatrix} -2 & -2 & 0 & 0 & 0 & 2 & 0 & 2 \\ -1.5 & 0.5 & -1.5 & 0.5 & -0.5 & -0.5 & 1.5 & 1.5 \end{pmatrix}$$

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- Cov$(\bar{x}_1, \bar{x}_1) = \frac{1}{7} \sum_{k=1}^{8} \bar{x}_1^k \cdot \bar{x}_1^k = \frac{1}{7}(4 + 4 + 4 + 4) = \frac{16}{7}$

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  $\frac{1}{7}((-1.5)^2+0.5^2+(-1.5)^2+0.5^2+(-0.5)^2+(-0.5)^2+1.5^2+1.5^2) = \frac{10}{7}$

# Example

$$\mathsf{Cov}(\bar{X}) = \begin{pmatrix} \frac{16}{7} & \frac{4}{7} \\ \frac{4}{7} & \frac{10}{7} \end{pmatrix}$$

# Eigenvector and eigenvalue

Given a square matrix A, an eigenvalue $\lambda$ and its associated eigenvector $\mathbf{v}$ are, by definition, a pair obeying the relation

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Equivalently,

$$(A - \lambda I)\mathbf{v} = 0$$

where $I$ is the identity matrix, implying

$$\mathrm{d}et(A - \lambda I) = 0.$$

# Principal component analysis

- Each non-zero eigenvalue of the covariance matrix indicates the portion of the variance that is correlated with each eigenvector.

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- The second principal component corresponds to the same concept after all correlation with the first principal component has been subtracted out from the points.
- Thus, the sum of all the eigenvalues is equal to the sum squared distance of the points with their mean. PCA essentially rotates the set of points around their mean in order to align with the first few principal components. This moves as much of the variance as possible (using a linear transformation) into the first few dimensions.

# Example

$$\mathsf{Cov}(\bar{X}) = \begin{pmatrix} \frac{16}{7} & \frac{4}{7} \\ \frac{4}{7} & \frac{10}{7} \end{pmatrix}$$

# Eigenvalue computation for $2 \times 2$ matrices

The eigenvalues of a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be obtained by the characteristic polynomial

$$\mathrm{d}et \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc = \lambda^2 - (a+d)\lambda + (ad-bc)$$

with solutions

$$\lambda = \frac{a+d}{2} \pm \sqrt{\frac{(a+d)^2}{4} + bc - ad} = \frac{a+d}{2} \pm \frac{\sqrt{4bc + (a-d)^2}}{2}.$$

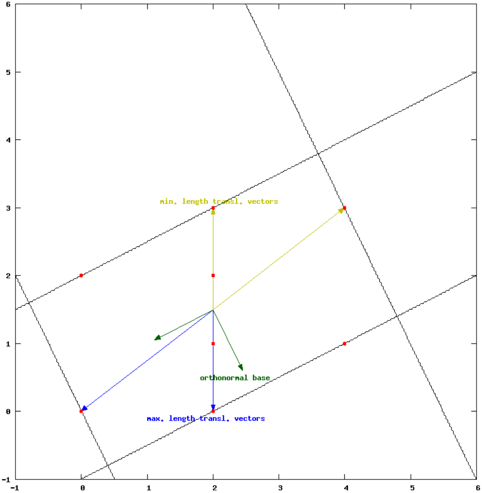
# Example

$$\mathsf{Cov}(\bar{X}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{16}{7} & \frac{4}{7} \\ \frac{4}{7} & \frac{10}{7} \end{pmatrix}$$

$$\lambda = \frac{a+d}{2} \pm \frac{\sqrt{4bc + (a-d)^2}}{2} = \frac{13}{7} \pm \frac{5}{7}$$

$$\lambda_1 = \frac{18}{7}$$

$$\lambda_2 = \frac{8}{7}$$


min. length transl. vectors
orthonormal base
max. length transl. vectors
6
5
4
3
2
1
0
-1
-1
0
1
2
3
4
5
6